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THE EFFECT OF INTERNAL CRACKS ON THE
DISTRIBUTION OF STRESS IN
THIN ELASTIC STRIPS AND CYLINDERS

by

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THE EFFECT OF INTERNAL CRACKS ON THE DISTRIBUTION OF STRESS IN THIN ELASTIC STRIPS AND CYLINDERS

Lectures given in the Department of Mathematics, North Carolina State College, Raleigh, North Carolina in April, 1963,

by

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PREFACE

This report is based on five lectures given by Dr. I. N. Sneddon at North Carolina State in April, 1963. The research reported here concerns certain crack problems in the mathematical theory of elasticity. A group of these problems were presented to us by Dr. George Irwin of NRL at the beginning of this sponsored research work; the present report concerns the solution of some of these. Part of the research results presented in this report have already been submitted in the form of three papers for possible publication.

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John W. Cell
Project Director

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1. Introduction.

The present report is based on a series of five lectures given in the Mathematics Department of North Carolina State College in April 1963, in which there was presented a connected account of some recent researches in the classical (infinitesimal) theory of elasticity. In particular the work done by the author and three collaborators R. P. Srivastav, R. J. Tait and J. L. Welch in the University of Glasgow during the preceding year is discussed in some detail.

In §2 there is a discussion of the two-dimensional problem of determining distribution of stress in a very long strip of uniform width $2c$ which has a Griffith crack of length $2a$ situated symmetrically in its interior. The stress field is supposed to be set up by the application of a known pressure to the inner surfaces of the crack. The equations of plane strain are used throughout (but the plane stress case can be derived by a trivial change in the values of the elastic constants). Two different kinds of boundary value problem are considered.

In Problem (a) (treated in §2.1) we assume that the surfaces of the strip are constrained in such a way that the normal component of the surface displacement and the surface shearing stress both vanish; this problem is exactly that of determining the distribution of stress in an infinite two-dimensional elastic medium containing an infinite row of identical Griffith cracks equally spaced. This boundary value problem is solved by reducing it to the solution of a pair of dual series relations which is in turn shown to be equivalent to that of an integral equation which can be solved easily for an arbitrary distribution of internal pressure. In §2.2 the solution in the case where the internal pressure is constant is considered in more detail; the critical value of the internal pressure is calculated using Griffith's criterion [formulae (2.35) and (2.36) below] and the effect of the fact that the strip is of finite width is illustrated by calculating the variation with a/c of the percentage change in the value of the critical pressure from the value in the 'infinite' case $c \gg a$ (Table 1 and Fig. 2 below). The shape of the crack and the variation of the normal component of stress across the line of the crack are also shown for some non-zero values of the ratio a/c (Figs. 4 and 5 below).

In Problem (b) (considered in §2.2) it is assumed that the surfaces of the strip are free from applied stress. Here the analysis is more complicated; the problem is again reduced to that of solving an integral equation - in this instance a Fredholm equation of the second kind - but the equation does not appear to have an exact (analytical) solution so that in any given case it would have to be solved numerically.

The discussion in §3 is an immediate generalization to the case of axial symmetry of the methods used in §2 so that the problem is now that of determining the distribution of stress in a very long circular cylinder which has an internal penny-shaped crack which has its centre lying on the axis of the cylinder and its plane normal to that axis. The stress field in the cylinder is due to the application of pressure to the surfaces of the crack. Again two kinds of problem are considered.

In Problem (a) (discussed in §3.1 and by an alternative method in §3.2) it is supposed that the shearing stress on the surface of the cylinder and that the radial component of the surface displacement are both identically zero. For an arbitrary (but axisymmetric) distribution of pressure on the crack surface the determination of the various quantities of physical interest is made to depend on the solution of a Fredholm integral of the second kind. An iterative solution of this equation with the free term corresponding to a constant internal pressure and valid for small values of the ratio a/c is given in §3.3; for values of a/c lying between 1 and 2 it is necessary to solve the question numerically and numerical values of the unknown function corresponding to a set of values of a/c are reported in the same section. From this solution it is a simple matter to calculate the variation with a/c of the critical pressure required to cause the crack to spread and that of the stress intensity factor introduced by Irwin; these are shown in Figs. 9 and 10 respectively. The results so obtained are compared (in Figs. 11 and 12) with those obtained in the analogous case in plane strain (§2.1 above) and it is shown that as far as an "engineering" approximation is concerned the size effect in the axisymmetric case can be simply gauged from that in the plane strain case.

Finally in Problem (b) (treated in §3.4) it is assumed that the curved surface of the cylinder is free from applied stress. Again the problem of determining the quantities of physical interest corresponding to an arbitrary axisymmetric distribution of stress on the crack surfaces is reduced to that of solving a Fredholm integral equation of the second kind but now the actual calculations are more complicated since the kernel of the integral equation is a function of the Poisson's ratio of the material of the cylinder. Again an iterative solution and numerical solutions for a constant internal pressure (and for Poisson's ratio equal to $\frac{1}{4}$) are given and the variation with a/c of the critical pressure and the Irwin stress intensity factor calculated (Cf. Figs. 16, 17). These two diagrams also afford a comparison between the solutions of Problems (a) and (b); if $a < \frac{1}{2} c$ there is little difference between the size effects in the two cases but if $a > \frac{1}{2} c$ the effect is more pronounced in the case in which the radial component of the surface displacement of the cylinder is zero.

2. Distribution of Stress in a Strip of Finite Width containing a Griffith crack.

We begin by considering the distribution of stress in a very long strip of uniform width 2c which has a Griffith crack of length 2a in its interior. We shall assume that the crack is perpendicular to the edges of the strip and that its centre lies on the central line of the strip (Cf. Fig. 1) and that the state of stress in the strip is due to the application of pressure of prescribed value to the surfaces of the crack. In addition we shall make the usual assumptions of the classical (infinitesimal) theory of elasticity and, in particular, we shall solve the equations corresponding to a state of plane strain in the strip. The results in the case of plane stress can easily be deduced by a trivial change in the values of the elastic constants. We employ the notation of Green and Zerna (1954).

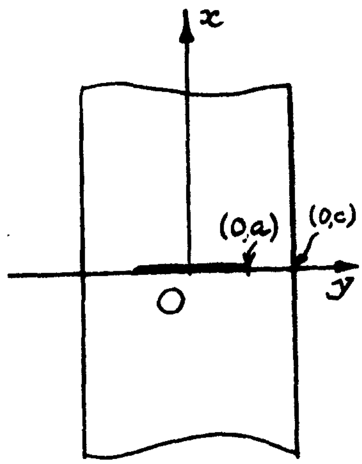

Fig. 1

We shall consider two types of boundary value problem. In problem (a) we assume that the surfaces of the strip are constrained in such a way that the normal component of the surface displacement and the shearing stress both vanish. This problem has been considered by Westergaard (193) and Green and England (1963) - in the form of a discussion of the state of stress in an infinite thin plate containing an infinite row of identical Griffith cracks evenly spaced; the solution given here is that derived recently by Sneddon and Srivastav (1963) which gives the results in a form suitable for numerical calculation and has the added advantage that it can be generalized to provide the solution of the analogous problem in three dimensions (Cf. §3.2 below). In problem (b) we assume that the surfaces of the strip are free from stress. This problem does not seem to have been discussed previously; again we follow the method recently devised by Sneddon and Srivastav (1963) which is a generalization to the elastostatic case of a method used by Sneddon (1962) in the solution of a simple problem in electrostatics.

2.1. Solution of Problem (a).

In the discussion of the first of these two problems we take our unit of length to be such that the width of the strip is 2π (i.e., in the notation of Fig. 1 we take $c = \pi$). The problem of determining the stress in the neighbourhood of the crack is then obviously equivalent to that of determining the stress in the semi-infinite strip $-\pi \leqslant y \leqslant \pi$, $x \geqslant 0$ when the boundary $x = 0$ is subjected to the conditions

$$\sigma_{xy} = 0, \qquad -\pi \leqslant y \leqslant \pi, \tag{2.1}$$

$$\sigma_{xx} = -2\mu f(y), \qquad -a < y < a, \tag{2.2}$$

$$u_x = 0, \qquad a < |y| \leqslant \pi \tag{2.3}$$

where μ denotes Lamé's constant, and the function $f(y)$ is prescribed. If we assume that the edges of the strip are constrained in such a way that the normal component of the displacement and the shearing stress both vanish then we must ensure that when $y = \pm\pi$,

$$\sigma_{xy} = u_y = 0, \qquad 0 < x < \infty. \tag{2.4}$$

We further assume that as $x \rightarrow \infty$ the components of stress and displacement all tend to zero. We shall satisfy this condition by finding a solution which satisfies the conditions

$$\sigma_{xy} = u_x = 0, \qquad -\pi \leqslant y \leqslant \pi \tag{2.5}$$

on the line $x = \delta$ and then finding its limiting form as $\delta \rightarrow \infty$

If we assume expressions of the form

$$u_x = \tfrac{1}{2} u_o\left(1 - \frac{x}{\delta}\right) - \sum_{n=0}^{\infty} a_n \operatorname{cosech}(n\delta)\Big\{n(\delta - x)\cosh n(\delta - x) -$$

$$- (2 - 2\eta + n\delta\coth n\delta)\sinh n(\delta - x)\Big\}\cos(ny) \qquad (2.6)$$

$$u_y = -\sum_{n=1}^{\infty} a_n \operatorname{cosech}(n\delta)\Big\{n(\delta - x)\sinh n(\delta - x) + (1 - 2\eta - n\delta\coth n\delta)\times$$

$$\times \cosh n(\delta - x)\Big\}\sin(ny) \qquad (2.7)$$

for the components of the displacement then (if we are considering plane strain) the components of stress are given by the equations

$$\frac{\sigma_{xx}}{2\mu} = -\frac{(1-\eta)u_o}{2(1-2\eta)\delta} - \sum_{n=1}^{\infty} n\,a_n \operatorname{cosech}(n\delta)\Big\{(1 + n\delta\coth n\delta)\times$$

$$\times \cosh n(\delta - x) - n(\delta - x)\sinh n(\delta - x)\Big\}\cos ny$$

$$\frac{\sigma_{yy}}{2\mu} = -\frac{\eta u_o}{2(1-2\eta)\delta} - \sum_{n=1}^{\infty} n\,a_n \operatorname{cosech}(n\delta)\Big\{(1 - n\delta\coth n\delta)\times$$

$$\times \cosh n(\delta - x) + n(\delta - x)\sinh n(\delta - x)\Big\}\cos(ny)$$

$$\frac{\sigma_{xy}}{2\mu} = \sum_{n=1}^{\infty} n^2 a_n \operatorname{cosech}(n\delta)\Big\{(\delta - x)\cosh n(\delta - x) - \delta\coth n\delta\,\times$$

$$\times \sinh n(\delta - x)\Big\}\sin(ny)$$

where η denotes Poisson's ratio.

This solution obviously satisfies the condition (2.1) on $x = 0$, the conditions (2.4) on $y = \pm \pi$, and the conditions (2.5) on $x = \delta$. Also on $x = 0$ we have the expressions

$$u_x = 2(1 - \eta)\left\{\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos ny\right\}$$

$$\frac{\sigma_{xx}}{2\mu} = -\tfrac{1}{2}\alpha a_0 - \sum_{n=1}^{\infty} n a_n \left\{1 + k(n\delta)\right\} \cos ny$$

where

$$a_0 = \frac{u_0}{2(1 - \eta)}, \qquad \alpha = \frac{2(1 - \eta)^2}{(1 - 2\eta)\delta} \tag{2.8}$$

and

$$k(\xi) = \frac{\xi + 2e^{-\xi} \sinh\xi}{\sinh^2 \xi}. \tag{2.9}$$

Hence we shall have solved the relevant boundary value problem if we can find a sequence of constants $\left\{a_n\right\}$ satisfying the dual series relations

$$\tfrac{1}{2}\alpha a_0 + \quad n a_n \left\{1 + k(n\delta)\right\} \cos(ny) = f(y), \quad 0 \leqslant y < a, \tag{2.10}$$

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(ny) = 0, \qquad a < y \leqslant \pi. \tag{2.11}$$

If we now consider the case of a long strip so that $\delta \gg \pi$ we find that equations (2.6) and (2.7) reduce to

$$u_x = \tfrac{1}{2} u_0\left(1 - \frac{x}{\delta}\right) + \sum_{n=1}^{\infty} \left\{a_n \; 2(1 - \eta) + nx\right\} e^{-nx} \cos(ny), \tag{2.12}$$

$$u_y = - \sum_{n=1}^{\infty} a_n \left\{(1 - 2\eta) - nx\right\} e^{-nx} \sin(ny), \tag{2.13}$$

and that the corresponding components of stress are given by the equations

$$\frac{\sigma_{xx}}{2\mu} = -\tfrac{1}{2}\alpha a_0 - \sum_{n=1}^{\infty} n a_n (1 + nx) e^{-nx} \cos(ny), \tag{2.14}$$

$$\frac{\sigma_{yy}}{2\mu} = -\tfrac{1}{2}\beta a_0 - \sum_{n=1}^{\infty} n a_n (1 - nx) e^{-nx} \cos(ny), \tag{2.15}$$

$$\frac{\sigma_{xy}}{2\mu} = - x \sum_{n=1}^{\infty} n^2 a_n e^{-nx} \sin(ny), \tag{2.16}$$

where the constant β is defined by the equation

$$\beta = \frac{2\eta(1 - \eta)}{(1 - 2\eta)\delta}$$

and the constants $\{a_n\}$ satisfy the dual series relations

$$\tfrac{1}{2}\alpha a_0 + \sum_{n=1}^{\infty} n a_n \cos(ny) = f(y), \quad 0 \leq y < a, \tag{2.17}$$

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(ny) = 0, \quad a < y \leq \pi. \tag{2.18}$$

In the case of an infinitely long strip we may take $\alpha = 0$.

To solve the pair of dual series equations (2.17) and (2.18) we make use of a method due to Srivastav (1963). We make the assumption that when $0 \leq y < a$

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos(ny) = \cos(\tfrac{1}{2} y) \int_y^a \frac{g(t)\,dt}{\sqrt{(\cos y - \cos t)}} \tag{2.19}$$

and reduce the problem to that of determining the function $g(t)$. From the theory of Fourier series it follows that

$$a_0 = \sqrt{2} \int_0^a g(t)\,dt, \quad a_n = \frac{1}{\sqrt{2}} \int_0^a g(t)\left[P_n(\cos t) + P_{n-1}(\cos t)\right] dt \tag{2.20}$$

where P_n denotes the Legendre polynomial of degree $\underline{n}$.

Now if $\alpha = 0$ in equation (2.17) we may integrate both sides of the equation to obtain the relation

$$\sum_{n=1}^{\infty} a_n \sin ny = F(y), \quad 0 \leq y < a \tag{2.21}$$

where the function $F(y)$ is defined by the equation

$$F(y) = \int_0^y f(u)\,du. \tag{2.22}$$

If we substitute from equations (2.20) into the left hand side of equation (2.21) and interchange the order of integration and summation we find that equation (2.21) is equivalent to the integral equation

$$\int_0^a S(y, t) g(t)\,dt = F(y), \quad 0 \leq y < a \tag{2.23}$$

where the kernel $S(y, t)$ is defined by the relation

$$S(y, t) = \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \left[P_n(\cos t) + P_{n-1}(\cos t)\right] \sin(ny) \tag{2.24}$$

It is easily shown that the function

$$S(y,\ t) = \frac{\cos(\frac{1}{2}\ y)H(y - t)}{\sqrt{(\cos t - \cos y)}} \tag{2.25}$$

has as its Fourier half-range sine series in the range $0 \leqslant y < \pi$ the expansion on the left hand side of equation (2.24). Substituting this expression for the kernel into the integral equation (2.23) we find that it reduces to the form

$$\int_0^y \frac{g(t)\ \cos(\frac{1}{2}\ y)\, d\, t}{\sqrt{(\cos\ t - \cos y)}} = F(y), \quad 0 \leqslant y < a < \pi \tag{2.26}$$

which can easily be shown to possess the solution

$$g(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{F(y)\sin(\frac{1}{2}\ y)\, d\, y}{\sqrt{(\cos\ y - \cos t)}}\ , \tag{2.27}$$

where $F(y)$ is given by equation (2.22).

From equation (2.12) we have that the surface displacement of the crack $u_x^{(o)}\ (y) = u_x(0,\ y)$ is given by the equation

$$u_x^{(o)}\ (y) = 2(1 - \eta)\left\{\tfrac{1}{2}\, a_0 + \sum_{n=1}^{\infty} a_n \cos(ny)\right\},\quad 0 \leqslant\ y < a$$

so that it follows from equation (2.19) that

$$u_x^{(o)}\ (y) = 2(1 - \eta)\ \cos(\tfrac{1}{2}\ y)\int_y^a \frac{g(t)\, d\, t}{\sqrt{(\cos\ y - \cos\ t)}}. \tag{2.28}$$

Similarly the normal component of stress across the line of the crack $\sigma^{(o)}_{xx}(y) = \sigma_{xx}(0, y)$ is given by the equation

$$\sigma^{(o)}_{xx}(y) = -2\mu \sum n\, a_n \cos(ny) = -2\mu \frac{\partial}{\partial y}\left\{\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \sin(ny)\right\}$$

and substituting from equation (2.19) we find that this is equivalent to the equation

$$\sigma^{(o)}_{xx}(y) = -2\mu \frac{\partial}{\partial y}\left\{\cos(\tfrac{1}{2} y) \int_0^y \frac{g(t)\, dt}{\sqrt{(\cos y - \cos t)}}\right\}. \qquad (2.29)$$

The other physical quantity in which we are interested is the strain energy of the crack in the case in which the normal stress across the surface of the crack is constant . This is easily shown to be given by the equation

$$W = -2 \int_0^a \sigma^{(o)}_{xx}(y) u^{(o)}_x(y) dy$$

If we substitute the expression (2.28) for $u^{o}_{x}(y)$ into this equation and interchange the order of the integrations we find that

$$W = \frac{4\sqrt{2}(1 - \eta^2)p_0^2}{E} \int_0^a \tan(\tfrac{1}{2} t)\, dt \int_0^t \frac{\cos(\tfrac{1}{2} u)\, du}{\sqrt{(\cos u - \cos t)}}. \qquad (2.30)$$

2.2. Crack opened out by constant internal pressure.

If the crack is opened out by a constant internal pressure p_0, then, in the notation of equation (2.2) we take $f(y) = p_0/2\mu$. It follows from equation (2.27) that in this case we need to evaluate dI/dt where

$$I = \int_0^t \frac{x \sin(\frac{1}{2} x)\, dx}{\sqrt{(\cos x - \cos t)}}$$

or, what is the same thing,

$$\frac{dI}{dt} = - \tfrac{1}{2} \sin(\tfrac{1}{2} t) \frac{dI}{d\tau} \quad ,$$

where $\tau = \cos \frac{1}{2} t$ and

$$I = 2\sqrt{2}\int_\tau^1 \frac{(\cos^{-1} z) dz}{\sqrt{(z^2 - \tau^2)}} \; .$$

Using the formula for integrating by parts we find that

$$I = 2\sqrt{2}\left\{ \int_\tau^1 \frac{\log\left[z + \sqrt{(z^2 - \tau^2)}\right] dz}{\sqrt{(1 - z^2)}} = \cos^{-1}\tau \, . \, \log \tau \right\}$$

from which it follows that

$$\frac{dI}{d\tau} = - \frac{\sqrt{2}\pi}{\tau} = - \sqrt{2}\,\pi \sec(\tfrac{1}{2} t)$$

and hence that

$$\frac{dI}{dt} = \frac{\pi}{\sqrt{2}} \tan(\tfrac{1}{2} t).$$

From equation (2.27) we can therefore deduce that

$$g(t) = \frac{p_0}{\sqrt{2}\,\mu} \tan(\tfrac{1}{2} t). \tag{2.31}$$

The surface displacement $u_x^{(o)}(y)$ of the crack is then given by equation (2.28) in the form

$$u_x^{(o)}(y) = \frac{2\sqrt{2}(1-\eta^2)p_o}{E}\cos(\tfrac{1}{2}y)\int_y^a \frac{\tan(\frac{1}{2}t)\,dt}{\sqrt{(\cos y - \cos t)}} .$$

The integration is elementary and it is readily shown that

$$u_x^{(o)}(y) = \frac{4(1-\eta^2)p_o}{E}\log\left\{\frac{\cos(\frac{1}{2}y) + \sqrt{(\cos^2\frac{1}{2}y - \cos^2\frac{1}{2}a)}}{\cos\frac{1}{2}a}\right\}$$

$$|y| < a \qquad (2.32)$$

Similarly the normal component of stress across the line of the crack is given by equation (2.29) in the form

$$\sigma_{xx}^{(o)}(y) = -2p_o\frac{\partial}{\partial y}\left\{\cos\tfrac{1}{2}y\int_o^a \frac{\frac{1}{2}\sec(\frac{1}{2}t)\tan(\frac{1}{2}t)\,dt}{\sqrt{(\sec^2\frac{1}{2}y - \sec^2\frac{1}{2}t)}}\right\},$$

$$a < |y| \leq \pi$$

$$= -2p_o\frac{\partial}{\partial y}\left[\sin^{-1}(\sec\tfrac{1}{2}t\cos\tfrac{1}{2}y)\right]_{t=o}^{t=a}$$

$$= p_o\left\{\frac{\sin(\frac{1}{2}y)}{\sqrt{(\cos^2\frac{1}{2}a - \cos^2\frac{1}{2}y)}} - 1\right\}, \quad a < |y| \leq \pi . \qquad (2.33)$$

In a similar way equation (2.30) yields the formula

$$W = \frac{4\sqrt{2}(1-\eta^2)p_o^2}{E}\int_o^a \tan(\tfrac{1}{2}t)\,dt\int_o^t \frac{\cos(\frac{1}{2}y)\,dy}{\sqrt{(\cos y - \cos t)}}$$

for the elastic energy of the crack. Now

$$\int_0^t \frac{\cos(\frac{1}{2}y)\,dy}{\sqrt{(\cos y - \cos t)}} = \frac{1}{\sqrt{2}} \int_0^t \frac{\cos(\frac{1}{2}y)\,dy}{\sqrt{(\sin^2\frac{1}{2}t - \sin^2\frac{1}{2}y)}} = \frac{\pi}{\sqrt{2}}$$

so that in our system of units

$$W = \frac{4(1-\eta^2)\pi p_0^2}{E} \int_0^a \tan(\tfrac{1}{2}t)\,dt = \frac{8(1-\eta^2)p_0^2\pi}{E} \log(\sec\tfrac{1}{2}a)\,.$$

In conventional units, if $2c$ is the width of the strip

$$W = \frac{8(1-\eta^2)p_0^2\,c^2}{\pi E} \log\left(\sec\frac{\pi a}{2c}\right). \tag{2.34}$$

If the surface tension of the material is T, the surface energy of the crack is $U = 4aT$ and Griffith's criterion

$$\frac{\partial}{\partial a}(W - U) = 0$$

for the critical value p_{cr} of p_0 when the crack length is $2\underline{a}$ leads to the equation

$$p_{cr} = \left\{\frac{ET}{(1-\eta^2)c}\cot\frac{a\pi}{2c}\right\}^{\frac{1}{2}}.$$

In the infinite case (i.e. $c = \infty$) the corresponding value is known to be

$$p_{cr}^{(\infty)} = \left\{\frac{2\,ET}{\pi(1-\eta^2)a}\right\}^{\frac{1}{2}}$$

so that we may write the last result in the fprm

$$p_{cr} = p_{cr}^{(\infty)}\,\omega\,(a/c) \tag{2.35}$$

where

$$\omega(z) = \left\{\tfrac{1}{2}\pi z \cot(\tfrac{1}{2}\pi z)\right\}^{\frac{1}{2}} . \tag{2.36}$$

If we denote by Π the percentage change in the value of the critical pressure p_{cr} from the value $p_{cr}^{(\infty)}$ corresponding to the infinite case ($c >> a$) so that

$$\Pi(a/c) = \frac{p_{cr}^{(\infty)} - p_{cr}}{p_{cr}^{\infty}} \times 100$$

then

$$\Pi(a/c) = 100\left\{1 - \left[\frac{\pi a}{2c}\cot\frac{\pi a}{2c}\right]^{-\frac{1}{2}}\right\}. \tag{2.37}$$

The variation of the functions ω and Π with a/c is shown in Table 1 and that of Π is shown graphically in Fig. 2.

We now return to the consideration of the shape of the crack and the value of the normal component of stress across the line of the crack. Putting y equal to zero in equation (2.32) we find that $u_x^{(o)}(0) = \epsilon$, the depth of the crack is given by the equation

$$\epsilon = \epsilon^{(\infty)} d(a/c) \tag{2.38}$$

where $\epsilon^{(\infty)}$ denotes the depth of the crack in the case $c >> a$ and is given by the equation

$$\epsilon^{(\infty)} = \frac{2(1-\eta^2)p_0 a}{E} \tag{2.39}$$

(in conventional units) and the function $\underline{d}(\underline{z})$ is defined by the relation

$$d(z) = \frac{2}{\pi z}\log\tan\left\{\tfrac{1}{4}\pi(z+1)\right\}. \tag{2.40}$$

Table 1: Variation of the functions ω and Π with a/c.

a/c	ω	Π
0.1	0.9958	0.42
0.2	0.9833	1.67
0.3	0.9617	3.83
0.4	0.9300	7.00
0.5	0.8862	11.37
0.6	0.8274	17.26
0.7	0.7489	25.11
0.8	0.6389	36.11
0.9	0.4732	52.68
0.92	0.4272	57.28
0.94	0.3735	62.65
0.96	0.3080	69.20
0.98	0.2256	77.44
0.99	0.1562	84.38

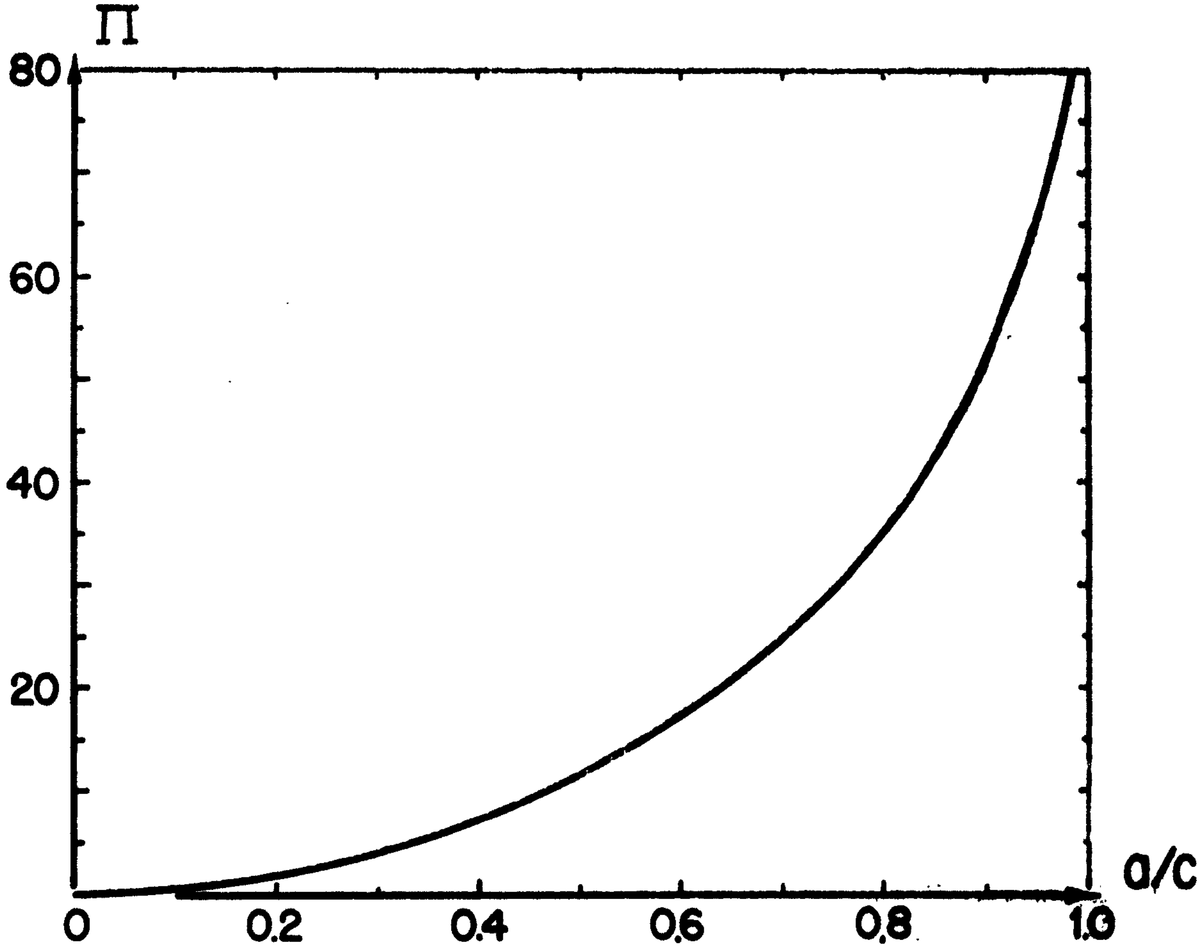

Figure 2

Fig. 2 The variation with a/c of π, the percentage increase in the critical pressure over its value in the case $c = \infty$, for constant internal pressure.

Values of $\underline{d}(\underline{z})$ for a range of values of $\underline{z}$ between 0 and 1 are listed in Table 2, and the variation with a/c of ϵ, the depth of the crack, for a fixed internal pressure p_0 is shown in Fig. 3.

Written in terms of conventional units equation (2.32) takes the form

$$u_x^{(o)}(y) = \epsilon^{(\infty)} \cdot \frac{2c}{\pi a}\left\{\log\left[\cos\left(\frac{\pi y}{2c}\right) + \sqrt{\left(\cos^2\frac{\pi y}{2c} - \cos^2\frac{\pi a}{2c}\right)}\right] - \right.$$

$$\left. - \log\cos\left(\frac{\pi a}{2c}\right)\right\}. \qquad (2.41)$$

The variation of the ratio $u_x^{(o)}(y)/\epsilon^{(\infty)}$ with y for four values of the ratio a/c is shown in Fig. 4; the curve corresponding to the value $a/c = 0$ is an ellipse

$$u_x^{(o)}(y) = \epsilon^{(\infty)}\,(1 - y^2/a^2)^{\frac{1}{2}}$$

and it can be seen from the curves of Fig. 4 that even a substantial increase in the ratio a/c does not appreciably affect the shape of the curve although, (as we should expect on physical grounds) the 'minor axis' increases by nearly 80% as a/c increases from 0 to 0.9.

Similarly, it follows from equation (2.33) that (in conventional units) the normal component of stress across the line of the crack is given by the equation

$$\sigma_{xx}^{(o)}(y) = p_0\left\{\sin\left(\frac{\pi y}{2c}\right)\left[\cos^2\left(\frac{\pi a}{2c}\right) - \cos^2\left(\frac{\pi y}{2c}\right)\right]^{-\frac{1}{2}} - 1\right\},$$

$$a < |y| < c. \qquad (2.42)$$

The form of the variation of $\sigma_{xx}^{(o)}(y)$ with y in three cases is shown in Fig. 5, (the same value of p_0 being chosen in each case). As we should expect, we find that the $\underline{\text{shape}}$ of the curve is much the same whatever the value of the ratio a/c but the intensity is greater the higher the value of a/c.

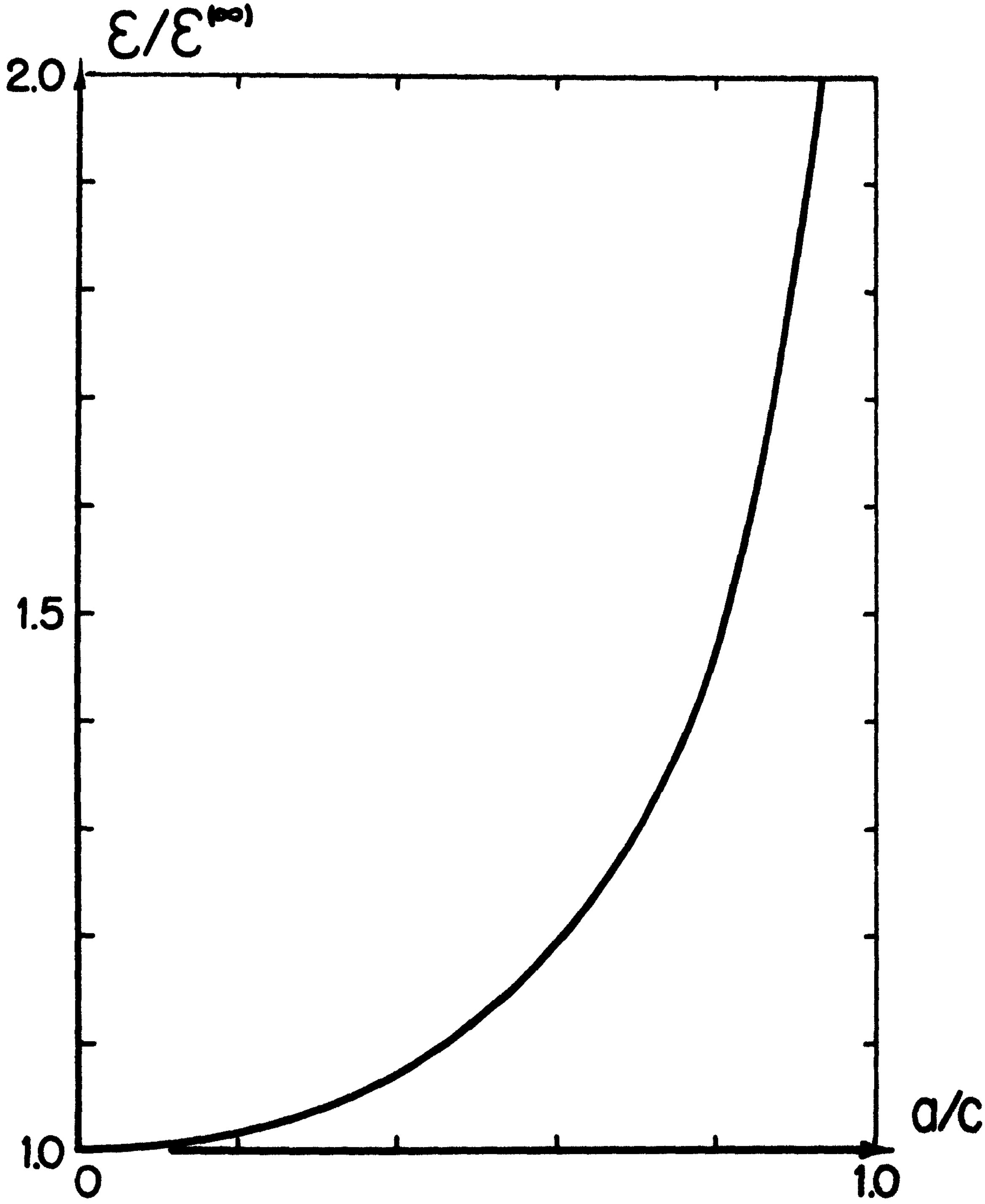

Figure 3

Fig. 3 The variation with a/c of the depth, ε, of the crack in the case of constant internal pressure p_o; $\varepsilon^{(\infty)} = 2(1-\eta^2)p_o a/E$ is the depth in the case a/c = 0.

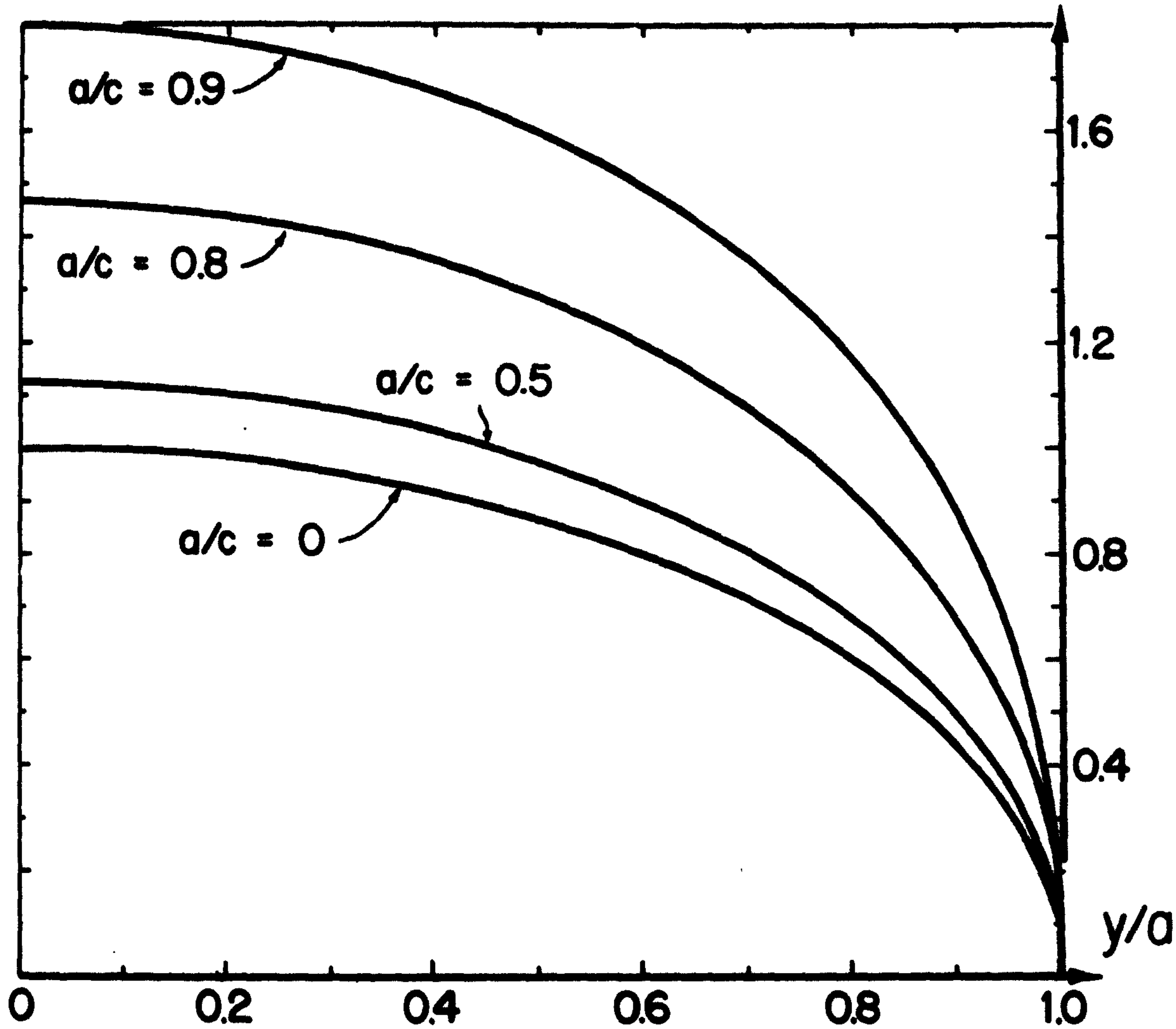

Figure 4

Fig. 4 The shape of the crack in the case of constant internal pressure p_o for four values of a/c. The unit on the vertical scale is $\varepsilon^{(\infty)}$ defined as in Fig. 3.

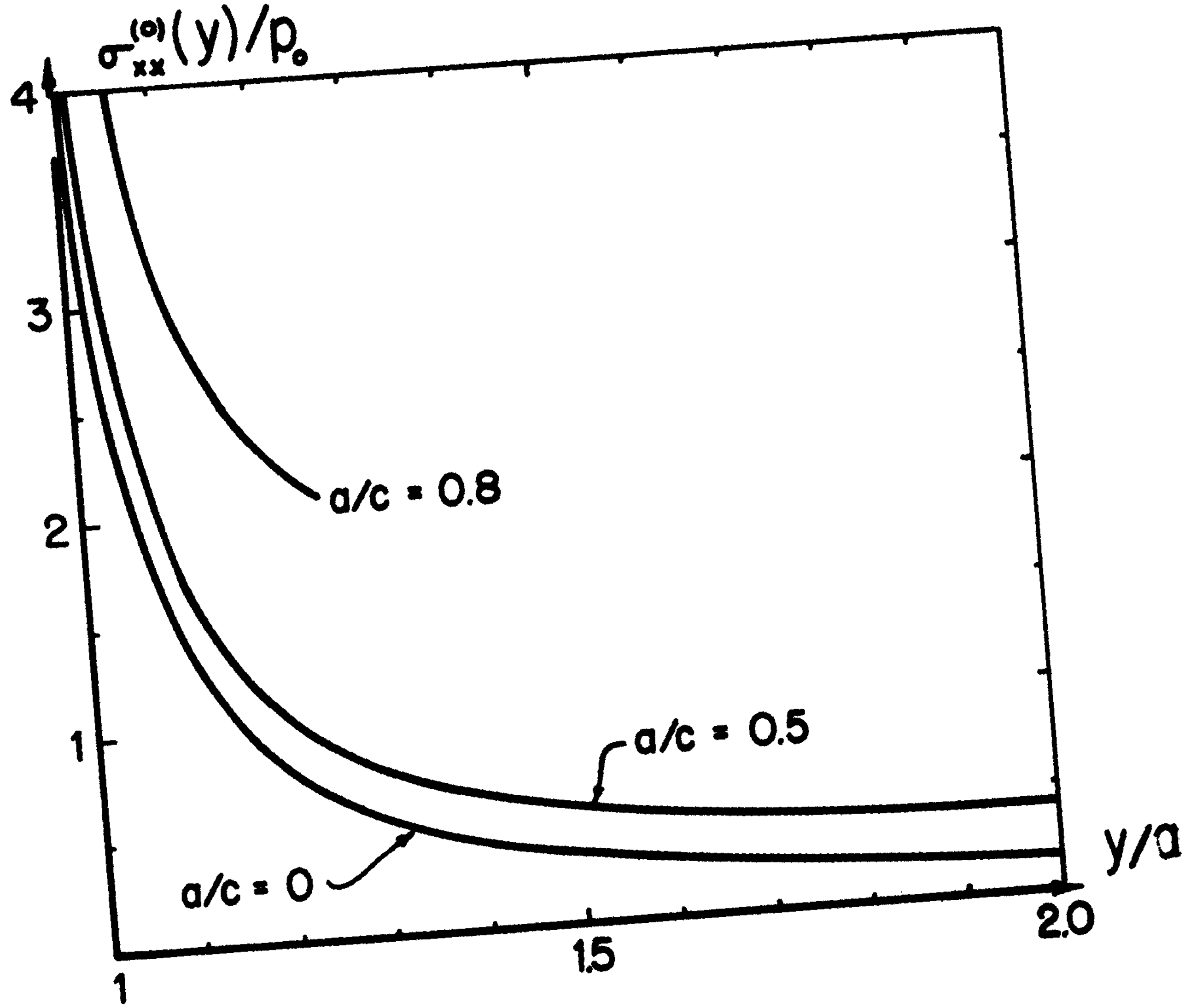

Figure 5

Fig. 5 The variation with y of $\sigma_{xx}^{(0)}(y)$ the normal component of stress across the line of the crack in the case of constant internal pressure p_o.

The <u>stress intensity factor</u> N is defined by the equation

$$N = \lim_{r \to 0+} \sqrt{r}\,\sigma_{xx}^{0}(a+r)$$

$$= p_0\left(\frac{c}{\pi}\tan\frac{\pi a}{2c}\right)^{\frac{1}{2}} .$$

If $c >> a$ N takes the value

$$N^{(\infty)} = p_0\left(\tfrac{1}{2}\,a\right)^{\frac{1}{2}}$$

so that we may write the percentage increase of N over $N^{(\infty)}$ as

$$\frac{N - N^{(\infty)}}{N^{(\infty)}} = n(a/c)$$

where

$$n(z) = \left\{\left[\frac{2}{\pi z}\tan(\tfrac{1}{2}\,\pi z)\right]^{\frac{1}{2}} - 1\right\} \times 100.$$

The function n(z) is tabulated in the second column of Table 2:

Table 2: Variation of the function d(z) with z.

z	d(z)	n(z)
0	1.0000	0.0
0.1	1.0041	1.0
0.2	1.0166	1.6
0.3	1.0393	4.0
0.4	1.0730	7.5
0.5	1.1223	12.9
0.6	1.1928	20.9
0.7	1.2975	33.7
0.8	1.4665	56.6
0.9	1.7982	112
0.92	1.9139	134
0.94	2.0685	160

2.3. Solution of Problem (b).

In the second problem the boundary conditions (2.1) - (2.3) remain but the conditions (2.4) are replaced by the conditions

$$\sigma_{xy} = \sigma_{yy} = 0, \quad y = \pm c, \quad 0 < x < \infty \tag{2.43}$$

If we denote by $\mathcal{F}_s$ and $\mathcal{F}_c$ the Fourier sine and cosine transform operators

$$\mathcal{F}_s\left\{\psi; \; x \rightarrow \xi\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \psi(x,y)\sin(\xi x)\,dx$$

$$\mathcal{F}_c\left\{\psi; \; x \rightarrow \xi\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \psi(x,y)\cos(\xi x)\,dx$$

then we can write the boundary conditions (2.43) in the equivalent forms

$$\mathcal{F}_s\left\{\left[\sigma_{xy}\right]_{y=c}; \; x \quad \xi\right\} = \mathcal{F}_c\left\{\left[\sigma_{yy}\right]_{y=c}; \; x \rightarrow \xi\right\} = 0 \tag{2.44}$$

We reduce the solution of this problem to that of a Fredholm integral equation of the second kind by a method similar to that employed recently by Sneddon (1962) in the solution of a boundary value problem of mixed type in electrostatics. We assume a stress field of the form

$$u_x = -\frac{1+\eta}{E}\sqrt{\frac{2}{\pi}}\int_0^\infty\left\{\left[f(\xi) + 2(1-\eta)g(\xi)\right]\cosh(\xi y) + \xi y g(\xi) \times \right.$$
$$\left. \times \sinh(\xi y)\right\}\xi^{-1}\sin(\xi x)\,d\xi + \frac{1+\eta}{E}\sqrt{\frac{2}{\pi}}\int_0^\infty \xi^{-1}\phi(\zeta) \times$$
$$\times\left[2(1-\eta) + \zeta x\right]e^{-\zeta x}\cos(\zeta y)\,d\zeta, \tag{2.45}$$

$$u_y = -\frac{1+\eta}{E}\sqrt{\frac{2}{\pi}}\int_0^\infty\left\{\left[f(\xi) - (1-2\eta)g(\xi)\right]\sinh(\xi y) + \xi y g(\xi) \times \right.$$

$$\left. \times\cosh(\xi y)\right\}\xi^{-1}\cos(\xi x)d\xi - \frac{1+\eta}{E}\sqrt{\frac{2}{\pi}}\int_0^\infty \zeta^{-1}\phi(\zeta) \times$$

$$\times\left[1 - 2\eta - \zeta x\right]e^{-\zeta x}\sin(\zeta y)\, d\zeta, \tag{2.46}$$

where E, as above, denotes Young's modulus, η denotes Poisson's ratio and f, g, and ϕ are functions to be determined from the boundary conditions. If we substitute these expressions into the stress-strain relations we find that the corresponding stress components are

$$\sigma_{xx} = \sqrt{\frac{2}{\pi}}\int_0^\infty \left\{(f + 2g)\cosh(\xi y) + \xi y g \sinh(\xi y)\right\}\cos(\xi x)d\xi$$

$$-\sqrt{\frac{2}{\pi}}\int_0^\infty \phi(\zeta)(1 + \zeta x)e^{-\zeta x}\cos(\zeta y)d\zeta, \tag{2.47}$$

$$\sigma_{yy} = - /\frac{2}{\pi}\int_0^\infty \left\{f\cosh(\xi y) + \xi y g\sinh(\xi y)\right\}\cos(\xi x)d$$

$$-\sqrt{\frac{2}{\pi}}\int_0^\infty \phi(\zeta)(1 - \zeta x)e^{-\zeta x}\cos(\zeta y)d\zeta, \tag{2.48}$$

$$\sigma_{xy} = \sqrt{\frac{2}{\pi}}\int_0^\infty\left\{(f + g)\sinh(\xi y) + \xi y g\cosh(\xi y)\right\}\sin(\xi x)d\xi$$

$$= \sqrt{\frac{2}{\pi}}\, x\int_0^\infty \zeta\phi(\zeta)\, e^{-\zeta x}\sin(\zeta y)d\zeta. \tag{2.49}$$

This solution therefore has the following properties:-

<u>On $x = 0$</u>: $\quad u_x = \dfrac{2(1-\eta^2)}{E}\sqrt{\dfrac{2}{\pi}}\displaystyle\int_0^\infty \zeta^{-1}\phi(\zeta)\cos(\zeta y)d\zeta$,

$$\sigma_{xx} = \sqrt{\frac{2}{\pi}}\int_0^\infty\left\{(f+2g)\cosh(\xi y) + \xi yg\sinh(\xi y)\right\}d\xi -$$

$$-\sqrt{\frac{2}{\pi}}\frac{d}{dy}\int_0^\infty \zeta^{-1}\phi(\zeta)\sin(\zeta y)d\zeta ,$$

$$\sigma_{xy} = 0;$$

<u>On $y = \pm c$</u>: $\mathcal{F}_c\left\{\sigma_{yy} \; ; \; x \rightarrow \xi\right\} = -f\cosh(\xi c) - \xi cg\sinh(\xi c) -$

$$-\frac{4\xi^2}{\pi}\int_0^\infty \frac{\zeta\phi(\zeta)\cos(\zeta c)d\zeta}{(\xi^2+\zeta^2)^2}$$

$$\mathcal{F}_s\left\{\sigma_{xy} \; ; \; x \rightarrow \xi\right\} = \pm\left\{(f+g)\sinh(\xi c) + \xi cg\cosh(\xi c)\right\} \mp$$

$$\mp\frac{4\xi}{\pi}\int_0^\infty \frac{\zeta^2\phi(\zeta)\sin(\zeta c)d\zeta}{(\xi^2+\zeta^2)^2} .$$

It follows from these equations that the boundary condition (2.1) is automatically satisfied and that the remaining conditions (2.2) and (2.3) on $x = 0$ are satisfied if we choose f, g, ϕ such that

$$\sqrt{\frac{2}{\pi}}\int_0^\infty\left\{(f+2g)\cosh(\xi y) + \xi yg\sinh(\xi y)\right\}d\xi -$$

$$-\sqrt{\frac{2}{\pi}}\frac{d}{dy}\int_0^\infty \zeta^{-1}\phi(\zeta)\sin(\zeta y)d\zeta = -p(y), \quad 0 \leq y < a, \tag{2.50}$$

$$\int_0^\infty \zeta^{-1}\phi(\zeta)\cos(\zeta y)d\zeta = 0, \quad a < y \leq c \tag{2.51}$$

where we have written $p(y)$ for $2\mu f(y)$. Similarly the conditions (2.44) on $y = \pm c$ will be satisfied if

$$f \cosh(\xi c) + \xi c\, g \sinh(\xi c) = -2 i_1(\xi) \tag{2.52}$$

$$(f + g) \sinh(\xi c) + \xi c\, g \cosh(\xi c) = 2 i_2(\xi) \tag{2.53}$$

where the functions $i_1(\xi)$, $i_2(\xi)$ are defined by the integrals

$$i_1(\xi) = \frac{2\xi^2}{\pi}\int_0^\infty \frac{\zeta\phi(\zeta)\cos(\zeta c)d\zeta}{(\xi^2 + \zeta^2)^2}, \quad i_2(\xi) = \frac{2\xi}{\pi} \quad \frac{\zeta^2\phi(\zeta)\sin(\zeta c)d\zeta}{(\xi^2 + \zeta^2)^2}. \tag{2.54}$$

If we make the representation

$$\phi(\zeta) = \zeta \int_0^a t\,\psi(t) J_0(\zeta t)\,dt \tag{2.55}$$

of the function $\phi(\zeta)$ then it is easily seen that the equation (2.51) is satisfied whatever the form of the function $\psi(t)$ but that equation (2.50) is satisfied only if

$$\frac{d}{dy}\int_0^y \frac{t\psi(t)\,dt}{\sqrt{(y^2 - t^2)}} - \int_0^\infty \left\{ (f + 2g)\cosh(\xi y) + \xi y\, g \sinh(\xi y)\right\} d\xi =$$

$$= \sqrt{\frac{\pi}{2}}\; p(y), \quad 0 \leqslant y < a.$$

If we integrate both sides of this equation with respect to y from 0 to $y(< a)$, we see that it is equivalent to the equation

$$\int_0^y \frac{t\psi(t)\,dt}{\sqrt{(y^2 - t^2)}} - \int_0^\infty \left\{\xi^{-1}(f + g)\sinh(\xi y) + y\, g \cosh(\xi y)\right\} d\xi =$$

$$= P(y), \quad 0 \leqslant y < a$$

where

$$P(y) = \sqrt{\frac{\pi}{2}} \int_0^y p(t)\,dt, \qquad 0 \le y < a. \tag{2.56}$$

Regarding this as an integral equation of Abel type for $\psi(t)$ in terms of known functions f, g, P we see that

$$t\psi(t) = \frac{2}{\pi}\int_0^\infty \xi^{-1}(f+g)\,d\xi \frac{d}{dt}\int_0^t \frac{y\sinh(\xi y)\,dy}{\sqrt{(t^2-y^2)}} +$$

$$+\frac{2}{\pi}\int_0^\infty g\,d\xi \frac{d}{dt}\int_0^t \frac{y^2\cosh(\xi y)\,dy}{\sqrt{(t^2-y^2)}} + \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{y\,P(y)\,dy}{\sqrt{(t^2-y^2)}}.$$

Using the results

$$\frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{y\sinh(\xi y)\,dy}{\sqrt{(t^2-y^2)}} = \xi t\, I_0(\xi t),$$

$$\frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{y^2\cosh(\xi y)\,dy}{\sqrt{(t^2-y^2)}} = t\left[I_0(\xi t) + \xi t\, I_1(\xi t)\right],$$

$$\frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{y\,P(y)\,dy}{\sqrt{(t^2-y^2)}} = \sqrt{\frac{2}{\pi}}\, t\int_0^t \frac{p(y)\,dy}{\sqrt{(t^2-y^2)}}$$

we find that the relation between ψ, f, g and p can be written in the form

$$\psi(t) = \int_0^\infty \left\{(f+2g)I_0(\xi t) + g\xi t\, I_1(\xi t)\right\} d\xi + $$

$$+\sqrt{\frac{2}{\pi}}\int_0^t \frac{p(y)\,dy}{\sqrt{(t^2-y^2)}} \tag{2.57}$$

On the other hand if we make the substitution (2.55) into the expressions (2.55) for the integrals $i_1(\xi)$, $i_2(\xi)$ we find that these integrals can be written in the form

$$i_1(\xi) = \xi \int_0^a u\,\psi(u)\, i_3\, du, \qquad i_2(\xi) = -\int_0^a u\,\psi(u)\,\frac{\partial i_3}{\partial c}\, du$$

where

$$i_3(\xi, u, c) = \frac{1}{\pi}\frac{\partial}{\partial\xi}\xi^2\int_0^\infty \frac{\cos(\zeta c) J_0(\zeta u)\, d\zeta}{\xi^2 + \zeta^2} .$$

The integral occurring on the right hand side can be evaluated by formula (14) on p.45 of vol.I of Erdelyi (1954) to give

$$i_3(\xi, u, c) = \frac{1}{2}\frac{\partial}{\partial\xi}\left\{\xi e^{-c\xi}\; I_0(\xi u)\right\} = \frac{1}{2}e^{-c\xi} \times$$

$$\times\left\{(1 - c\xi)\, I_0(\xi u) + \xi u\, I_1(\xi u)\right\}$$

so that

$$i_1(\xi) = \frac{1}{2}\xi\, e^{-c\xi}\int_0^a u\,\psi(u)\;\; (1 - c\xi) I_0(\xi u) + \xi u\, I_1(\xi u)\Big\}\, du \qquad (2.56)$$

$$i_2(\xi) = \frac{1}{2}\xi\; e^{-c\xi}\int_0^a u\,\psi(u)\left\{(2 - c\xi) I_0(\xi u) + \xi u\, I_1(\xi u)\right\} du \qquad (2.57)$$

Now if we solve equations (2.52) and (2.53) we find that

$$f(\xi) = -\frac{4}{2\xi c + \sinh(2\xi c)}\Big\{\left[\xi\, \underline{c}\cosh(\xi c) + \sinh(\xi c)\right] \times$$

$$\times\;\; i_1(\xi) + \xi c \sinh(\xi c)\, i_2(\xi)\Big\} \qquad (2.58)$$

$$g(\xi) = \frac{4}{2\xi c + \sinh(2\xi c)} \left\{ \sinh(\xi c) i_1(\xi) + \cosh(\xi c) i_2(\xi) \right\} \tag{2.59}$$

If we substitute from equations (2.56), (2.57) into equations (2.58), (2.59) and insert the resulting values of $f(\xi)$, $g(\xi)$ into equation (2.50) we find that the integral equation for the determination of the function $\psi(t)$ is

$$\psi(t) - \int_0^a \psi(u)\, K(t, u)\, du = q(t), \quad 0 \leqslant t < a \tag{2.60}$$

where the kernel $K(t, u)$ is defined by the equation

$$K(t, u) = u \int_0^\infty \frac{\xi \, d\xi}{2\xi c + \sinh(2\xi c)} \left\{ (2c^2\xi^2 - 6c\xi + 5 + 3e^{-2c\xi}) \times \right.$$

$$\times\, I_0(\xi u)\, I_0(\xi t) + (3 - 2c\xi + e^{-2c\xi}) \Big[\xi u\, I_0(\xi t) I_1(\xi u) +$$

$$\left. + \, \xi t\, I_0(\xi u)\, I_1(\xi t) \Big] + 2\xi^2 u t\, I_1(\xi u)\, I_1(\xi t) \right\} \tag{2.61}$$

and the free term is defined by the equation

$$q(t) = \sqrt{\frac{2}{\pi}} \int_0^t \frac{p(y)\, dy}{\sqrt{(t^2 - y^2)}} \tag{2.62}$$

In the case in which the internal pressure $p(y)$ is a constant p_0, say, we have $q(t) = (\frac{1}{2}\pi)^{\frac{1}{2}} p_0$, so that if we write

$$\psi(t) = (\tfrac{1}{2}\pi)^{\frac{1}{2}} p_0 \psi_1(t/c) \tag{2.63}$$

we find that $\psi_1(t)$ is the solution of the Fredholm equation

$$\psi_1(t) - \int_0^{\alpha} \psi_1(v)\, L(t, v)\, dv = 1, \quad 0 \leq t \leq \alpha \tag{2.64}$$

where $\alpha = a/c$ and

$$L(t, v) = c\, K(ct, cv)$$

$$= v \int_0^{\infty} \frac{\zeta\, d\zeta}{2\zeta + \sinh 2\zeta} \Big\{(2\zeta^2 - 6\zeta + 5 + 3e^{-2\zeta}) I_0(\zeta v) I_0(\zeta t)$$

$$+ (3 - 2\zeta + e^{-2\zeta})\Big[\zeta v\, I_0(\zeta t) I_1(\zeta v) + \zeta t\, I_0(\zeta v) I_1(\zeta t)\Big]$$

$$+ 2\zeta^2\, t v\, I_1(\zeta v) I_1(\zeta t)\Big\}. \tag{2.65}$$

An approximate solution of the integral equation (2.66) is

$$\psi_1(t) = 1 + \int_0^{\alpha} L(t, v)\, dv. \tag{2.66}$$

Using the results

$$\int_0^{\alpha} v\, I_0(\zeta v)\, dv = \frac{\alpha I_1(\zeta \alpha)}{\zeta}, \quad \int_0^{\alpha} v^2 I_1(\zeta v)\, dv = \frac{\alpha^2 I_2(\zeta \alpha)}{\zeta} \tag{2.67}$$

we find

$$\int_0^{\alpha} L(t, v)\, dv = \alpha \int_0^{\infty} \frac{d\zeta}{2\zeta + \sinh 2\zeta} \Big\{(2\zeta^2 - 6\zeta + 5 + 3e^{-2\zeta}) \times$$

$$\times\ I_1(\zeta\alpha)\, I_0(\zeta t) + (3 - 2\zeta + e^{-2\zeta}) \times$$

$$\times \Big[\zeta \alpha I_0(\zeta t) I_2(\zeta\alpha) + \zeta t\, I_0(\zeta\alpha) I_1(\zeta t)\Big] + 2\zeta^2 t \alpha I_2(\zeta\alpha) I_1(\zeta t)\Big\}. \tag{2.68}$$

3. Distribution of Stress in a Cylinder containing a Penny-Shaped Crack.

We shall now consider the distribution of stress in an infinitely long circular cylinder which has a penny-shaped crack in its interior with the centre of the crack lying on the axis of the cylinder and its plane perpendicular to that axis. (Cf. Fig. 6). We shall suppose that the deformation in the cylinder is produced by the application of the pressure to the surfaces of the crack. As in the two-dimensional case we shall consider two types of boundary conditions on the curved surface of the cylinder. In the first problem - denoted by problem (a) here - we suppose that the shearing stress on this surface and the radial component of the surface displacement both vanish; this problem has recently been discussed by Sneddon and Tait (1963) and we shall follow that solution here. In the second problem - called problem (b) below - we assume that the curved surface of the cylinder is free from stress; this problem has been considered recently by Collins (1962) and by Sneddon and Welch (1963). We shall outline the method of the latter paper here.

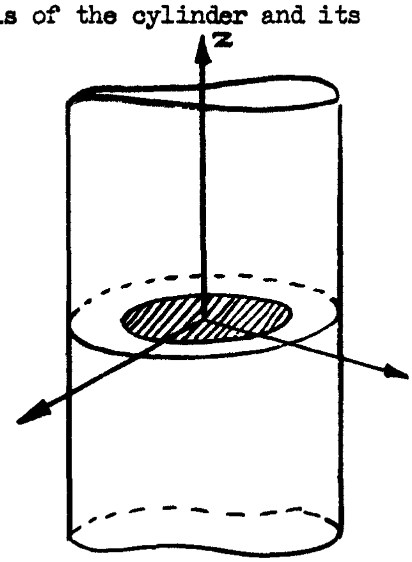

Fig.6

3.1. Solution of Problem (a).

If we take the radius of the crack to be our unit of length, and the radius of the cylinder to be c (> 1), then the problem of determining the distribution of stress in the neighbourhood of the crack is equivalent to that of finding the distribution of stress in the semi-infinite cylinder $\rho \leqslant c$, $z \geqslant 0$ when its plane boundary $z = 0$ is subjected to the conditions

$$\sigma_{\rho z} = 0, \qquad 0 \leqslant \rho \leqslant c, \qquad (3.1)$$

$$\sigma_{zz} = -2\mu f(\rho), \quad 0 \leqslant \rho < 1, \qquad (3.2)$$

$$u_z = 0, \qquad 1 < \rho \leqslant c \qquad (3.3)$$

where μ is Lamé's constant (rigidity modulus) and the function $f(\rho)$ is prescribed. If we assume that the cylindrical surface is kept fixed then on $\rho = c$ we must have

$$u_\rho = \sigma_{\rho z} = 0, \qquad 0 \leqslant z < \infty \tag{3.4}$$

It is well-known [Cf. Sneddon (1951), p.505] that a solution of the equations of elastic equilibrium in the axially symmetric equation is given by the equations

$$u_\rho = -\frac{1}{2\mu}\frac{\partial^2\chi}{\partial\rho\partial z}, \quad u_z = \frac{1}{2\mu}\left\{2(1-\eta)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right\}, \tag{3.5}$$

where $\chi(\rho, z)$ is an axisymmetric biharmonic function and η is Poisson's ratio. The components of the stress tensor can be determined from the stress-strain relations; we have

$$\sigma_{\rho\rho} = \frac{\partial}{\partial z}\left\{\eta\nabla^2\chi - \frac{\partial^2\chi}{\partial\rho^2}\right\}, \quad \sigma_{\rho z} = \frac{\partial}{\partial\rho}\left\{(1-\eta)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right\}$$

$$\sigma_{\phi\phi} = \frac{\partial}{\partial z}\left\{\eta\nabla^2\chi - \frac{1}{\rho}\frac{\partial\chi}{\partial\rho}\right\}, \quad \sigma_{zz} = \frac{\partial}{\partial z}\left\{(2-\eta)\nabla^2\chi - \frac{\partial^2\chi}{\partial z^2}\right\}.$$

A suitable type of biharmonic function for a problem of this type is defined by the equation

$$\chi = -2\mu\int_0^\infty \xi^{-2}\left\{\left[A(\xi) + 4(1-\eta)B(\xi)\right]I_0(\xi\rho) - \xi\rho B(\xi)I_1(\xi\rho)\right\}\sin(\xi z)\,d\xi$$
$$-2\mu\int_0^\infty \xi^{-3}F(\xi)(2\eta + \xi z)e^{-\xi z}J_0(\xi\rho)\,d\xi, \tag{3.6}$$

where A, B and F are functions of ξ alone. A solution of this form automatically satisfies equation (1.1). The corresponding expressions for σ_{zz} and u_z on the plane $z = 0$ are given respectively by the equations

$$\sigma_{zz} = -2\mu\int_0^\infty \xi\left\{\left[A(\xi) - 2\eta B(\xi)\right]I_0(\xi\rho) - \xi\rho B(\xi)I_1(\xi\rho)\right\}d\xi$$
$$-2\mu\int_0^\infty F(\xi)J_0(\xi\rho)\,d\xi, \tag{3.7}$$

$$u_z = 2(1-\eta)\int_0^\infty \xi^{-1}F(\xi)J_0(\xi\rho)\,d\xi. \tag{3.8}$$

From these last two equations it follows immediately that the boundary conditions (1.2) and (1.3) are satisfied if $A(\xi)$, $B(\xi)$, $F(\xi)$ satisfy the dual integral equations

$$\int_0^\infty F(\xi)J_0(\xi\rho)d\xi + \int_0^\infty \xi\left\{\left[A(\xi) - 2\eta B(\xi)\right]I_0(\xi\rho) - \xi\rho B(\xi)I_1(\xi\rho)\right\}d\xi = f(\rho), \quad 0 \leq \rho < 1,$$

$$\int_0^\infty \xi^{-1}F(\xi)J_0(\xi\rho)d\xi = 0, \qquad 1 < \rho \leq c.$$

It is known [Cf. Sneddon (1960)] that the second of these equations is automatically satisfied if $F(\xi)$ is written in terms of an unknown function $g(t)$ through the equations

$$F(\xi) = \xi\int_0^1 g(t)\sin(\xi t)dt, \quad g(0) = 0 \tag{3.9}$$

and that if we substitute this form into the first equation of the pair it reduces it to the relation

$$\int_0^\rho \frac{g'(t)dt}{\sqrt{(\rho^2 - t^2)}} = f(\rho) - \int_0^\infty \xi\left\{\left[A(\xi) - 2\eta B(\xi)\right]I_0(\xi\rho) - \xi\rho B(\xi)I_1(\xi\rho)\right\}d\xi$$

which can be thought of as an Abel type integral equation with solution

$$g(t) = h(t) - \frac{2}{\pi}\int_0^\infty \xi\left\{(A - 2\eta B)\int_0^t \frac{\rho I_0(\xi\rho)d\rho}{\sqrt{(t^2 - \rho^2)}} - \xi B\int_0^t \frac{\rho^2 I_1(\xi\rho)d\rho}{\sqrt{(t^2 - \rho^2)}}\right\}d\xi$$

where $h(t)$ is defined in terms of $f(\rho)$ by the equation

$$h(t) = \frac{2}{\pi}\int_0^t \frac{\rho f(\rho)\,d\rho}{\sqrt{(t^2 - \rho^2)}}. \tag{3.10}$$

Making use of the results

$$\int_0^t \frac{\rho I_0(\xi\rho)d\rho}{\sqrt{(t^2 - \rho^2)}} = \frac{\sinh(\xi t)}{\xi}, \quad \int_0^t \frac{\rho^2 I_1(\xi\rho)d\rho}{\sqrt{(t^2 - \rho^2)}} = \frac{\xi t\cosh(\xi t) - \sinh(\xi t)}{\xi^2}$$

we find that the relation connecting $g(t)$ with $A(\xi)$ and $B(\xi)$ may be written in the form

$$g(t) = h(t) - \frac{2}{\pi}\int_0^\infty\left\{\left[A(\xi) + (1 - 2\eta)B(\xi)\right]\sinh(\xi t) - \xi t B(\xi)\cosh(\xi t)\right\}d\xi. \tag{3.11}$$

Further, if we substitute from equation (3.9) into equation (3.8) we find that the normal component of the surface displacement is given by the equation

$$u_3^{\circ}(\rho) = 2(1-\eta)\int_{\rho}^{1}\frac{g(t)\,dt}{\sqrt{(t^2-\rho^2)}}, \qquad 0 \leqslant \rho \leqslant 1 \tag{3.12}$$

Another expression of physical interest is the energy W required to open out the crack. This is given by the equation

$$W = 4\pi\mu\int_0^1 \rho f(\rho)u_3^{\circ}(\rho)\,d\rho.$$

If we substitute the expression (3.12) for $u_3^{\circ}(\rho)$ and interchange the order in which we perform the integrations we find that

$$W = 4\pi^2\mu(1-\eta)\,\omega(c^{-1}) \tag{3.13}$$

where the function ω is defined by the equation

$$\omega(c^{-1}) = \int_0^1 h(t)g(t)dt. \tag{3.14}$$

The expression (3.13) for the energy W is derived on the assumption that the unit of length is the radius of the crack. If the length of the crack were <u>a</u> this expression would be modified by multiplying the right hand side by $\underline{a}^3$, i.e. we should have

$$W = 4\pi^2\mu(1-\eta)\,\omega(a/c)a^3. \tag{3.15}$$

In particular, if a constant pressure p_0 is applied to the crack, $f(\rho) = p_0/2\mu$ and so we have

$$h(t) = \frac{p_0 t}{\pi\mu}. \tag{3.16}$$

If we write

$$g(t) = \frac{p_0}{2\mu}\,\phi(t) \tag{3.17}$$

we find that

$$W = \frac{8p_0^2(1-\eta^2)}{3E}\,a^3\,\omega_1(a/c) \tag{3.18}$$

where

$$\omega_1(c^{-1}) = \frac{3}{2}\pi\int_0^1 t\,\phi(t)\,dt. \tag{3.19}$$

The Griffith criterion that the crack may spread is

$$\frac{\partial}{\partial a}(W - U) = 0 \tag{3.20}$$

where U, the surface energy of the crack, is given in terms of the surface tension T of the material by $U = 2\pi a^2 T$. This leads to the expression

$$p_{cr} = \left\{\frac{\pi E T}{2(1-\eta^2)a}\right\}^{\frac{1}{2}} \Omega_1(a/c) \tag{3.21}$$

for the critical value of the applied pressure to cause the crack to spread when its radius is a. In this equation the function $\Omega_1(x)$ is defined by the equation

$$\Omega_1(x) = \left\{\omega_1(x) + \tfrac{1}{3}x\,\omega_1'(x)\right\}^{-\frac{1}{2}}. \tag{3.22}$$

If $c >> a$, the critical pressure takes the value

$$p_{cr}^{(\infty)} = \left\{\frac{\pi E T}{2(1-\eta^2)a}\right\}^{\frac{1}{2}}$$

so that we may write equation (3.21) in the form

$$p_{cr} = p_{cr}^{(\infty)}\ \Omega_1(a/c). \tag{3.23}$$

Equation (3.11) gives one relation connecting the unknown functions $g(t)$, $A(\xi)$, $B(\xi)$ and the known function $h(t)$. The two remaining relations are given by the conditions on the curved surface $\rho = c$. It is easily shown that the values of $\sigma_{\rho z}$ and u_ρ on the surface $\rho = c$ corresponding to the form (3.6) for $\chi(\rho, z)$ are given by the equations

$$\left[\sigma_{\rho z}\right]_{\rho=c} = -2\mu z\int_0^\infty \zeta F(\zeta)e^{-\zeta z}J\,(\zeta c)\,d\zeta$$

$$- 2\mu z\int_0^\infty \xi\left\{\left[A(\xi) + 2(1-\eta)B(\xi)\right]I_1(\xi c) - \xi c B(\xi) I_0(\xi c)\right\} \times$$

$$\sin(\xi z)\,d\xi \tag{3.24}$$

$$[u_\rho]_{\rho=c} = -\int_0^\infty \zeta^{-1}F(\zeta)\ 1 - 2\eta - \zeta z\ \ J_1(\zeta c)e^{-\zeta z}\, d\zeta$$

$$+\int_0^\infty \Big\{\big[A(\xi) + 4(1-\eta)B(\xi)\big]I_1(\xi c) - \xi c B(\xi) I_0(\xi c)\Big\}\cos(\xi z)d\xi. \quad (3.25)$$

If we take the Fourier sine transform of both sides of equation (3.24) and make use of the fact, which follows from equations (3.4), that the Fourier sine transform of $[\sigma_{\rho z}]_{\rho=c}$ is identically equal to zero we obtain the equation

$$A(\xi)I_1(\xi c) - B(\xi)\big[c\xi I_0(c\xi) - 2(1-\eta)I_1(c\xi)\big] = i_2 - i_1\,, \quad (3.26)$$

where the functions $i_1(\xi)$, $i_2(\xi)$ are defined by the integrals

$$i_1(\xi) = \frac{4}{\pi}\ \ \frac{F(\zeta)J_1(\zeta c)}{\xi^2+\zeta^2}\, d\zeta\,, \quad i_2(\xi) = \frac{4\xi^2}{\pi}\int_0^\infty \frac{F(\zeta)J_1(\zeta c)d\zeta}{(\xi^2+\zeta^2)^2}\,. \quad (3.27)$$

Similarly if we take the Fourier cosine transform of both sides of equation (3.25) and make use of the fact that $[u_\rho]_{\rho=c}$ vanishes for all values of z, we obtain the equation

$$A(\xi)I_1(\xi c) - B(\xi)\big[c\xi I_0(c\xi) - 4(1-\eta)I_1(c\xi)\big] = i_2 - \eta i_1\,. \quad (3.28)$$

Solving the equations (3.26) and (3.28) we find that

$$A(\xi) = \frac{i_1(\xi)}{I_1(c\xi)}\left\{\frac{c\xi I_0(c\xi)}{2I_1(c\xi)} - (2-\eta)\right\} + \frac{i_2(\xi)}{I_1(c\xi)}\,, \quad B(\xi) = \frac{i_1(\xi)}{2I_1(c\xi)}. \quad (3.29)$$

Substituting from equation (3.9) into the first of equations (3.27) we find that $i_1(\xi)$ is related to $g(t)$ through the equation

$$i_1(\xi) = \frac{4}{\pi}\int_0^1 g(t)dt\int_0^\infty \frac{\zeta\sin(\zeta t)J_1(\zeta c)d\zeta}{\xi^2+\zeta^2}\,. \quad (3.30)$$

The inner integral can be derived easily from entry (5) on p.10 of Vol.2 of Erdelyi (1954) and we find that

$$i_1(\xi) = \frac{4}{\pi}\ \ K_1(c\xi)\int_0^1 g(u)\sinh(\xi u)du. \quad (3.31)$$

Similarly, it is easily shown that

$$i_2(\xi) = \frac{2}{\pi}\int_0^1 g(u)\Big\{\sinh(\xi u)\Big[c\,\xi K_0(c\,\xi) + K_1(c\,\xi)\Big] - \xi u\,\cosh(\xi u)K_1(c\xi)\Big\}du. \tag{3.32}$$

If we substitute from equations (3.31), (3.32) into equations (3.29) and thence into equation (3.11) we find that this last equation reduces to the integral equation

$$g(t) - \int_0^1 K(t, u)g(u)du = h(t) \tag{3.33}$$

in which the kernel $K(t, u)$ is defined by the equation

$$K(t, u) = \frac{4}{\pi^2}\int_0^\infty \left\{\frac{2\, I_1(c\,\xi)K_1(c\,\xi)}{I_1^2(c\,\xi)} \sinh(\xi t)\sinh(\xi u)\right.$$

$$\left. + \frac{\xi K_1(c\,\xi)}{I_1(c\,\xi)}\Big[u\,\cosh(\xi u)\sinh(\xi t) + t\,\cosh \xi t\,\sinh \xi u\Big]\right\}d\xi.$$

Now, using standard properties of Bessel functions (see Watson (1944) pp.79-80), we have that

$$\frac{d}{d\xi}\left[\frac{K_1(c\,\xi)}{I_1(c\,\xi)}\right] = c\;\frac{K_1'(c\,\xi)I_1(c\,\xi) - I_1'(c\,\xi)K_1(c\,\xi)}{I_1^2(c\,\xi)}$$

$$= -\tfrac{1}{2}\,c\;\frac{\big[I_0(c\,\xi) + I_2(c\,\xi)\big]K_1(c\,\xi) + \big[K_0(c\,\xi) + K_2(c\,\xi)\big]I_1(c\,\xi)}{I_1^2(c\,\xi)}$$

$$= -\frac{1}{\xi I_1^2(c\,\xi)}$$

so that

$$\int_0^\infty \frac{\sinh(\xi t)\,\sinh(\xi u)\,d\xi}{I_1^2(c\,\xi)} = -\int_0^\infty \frac{d}{d\xi}\left[\frac{K_1(c\,\xi)}{I_1(c\,\xi)}\right]\xi\,\sinh(\xi t)\sinh(\xi u)\,d\xi.$$

Integrating by parts, we can reduce this last term to

$$\int_0^\infty \frac{K_1(c\xi)}{I_1(c\xi)} \left\{ \sinh(\xi t)\sinh(\xi u) + \xi t \cosh(\xi t)\sinh(\xi u) + \right.$$

$$\left. + \xi u \cosh(\xi u)\sinh(\xi t) \right\} d\xi .$$

We therefore find that the kernel $K(t, u)$ of the integral equation (3.33) is given by the equation

$$K(t, u) = \frac{4}{\pi^2} \int_0^\infty \frac{K_1(c\xi)}{I_1(c\xi)} \sinh(\xi t)\sinh(\xi u)\, d\xi \tag{3.34}$$

which may be written in the form

$$K(t, u) = H\left(\frac{u+t}{c}\right) - H\left(\frac{u-t}{c}\right) \tag{3.35}$$

where the function $H(\lambda)$ is defined by the equation

$$H(\lambda) = \frac{2}{\pi^2 c} \int_0^\infty \frac{K_1(v)}{I_1(v)} \left[\cosh(\lambda v) - 1\right] dv. \tag{3.36}$$

3.2. Alternative Solution of Problem (a).

In view of the recent interest in the solution of dual series relations it is interesting to see that problem (a) can be reduced to that of solving a pair of such equations and that the use of a method of solution due to Sneddon and Srivastav (1963) leads to the same integral equation.

If we insert the biharmonic function

$$\chi(\rho, z) = -2\mu \sum_{n=1}^{\infty} \lambda_n^{-1} a_n (2\eta + \lambda_n z) e^{-\lambda_n z} J_0(\lambda_n \rho) \tag{3.37}$$

into the equations (3.5) we find that it corresponds to an axisymmetric displacement vector with components

$$u_\rho = -\sum_{n=1}^{\infty} \lambda_n a_n (1 - 2\eta - \lambda_n z) e^{-\lambda_n z} J_1(\lambda_n \rho), \tag{3.38}$$

$$u_z = 2\sum_{n=1}^{\infty} \lambda_n a_n(1 - \eta + \lambda_n z)e^{-\lambda_n z} J_0(\lambda_n \rho). \tag{3.39}$$

The $\underline{z}$-components of stress are then given by the equations

$$\sigma_{zz} = -2\mu\sum_{n=1}^{\infty} \lambda_n^2 a_n(1 + \lambda_n z)e^{-\lambda_n z} J_0(\lambda_n \rho), \tag{3.40}$$

$$\sigma_{z\rho} = -2\mu z \sum_{n=1}^{\infty} \lambda_n^3 a_n e^{-\lambda_n z} J_1(\lambda_n \rho). \tag{3.41}$$

It is an immediate consequence of equations (3.38) and (3.41) that the boundary conditions (3.4) will be satisfied provided that $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots$ are chosen to be the positive zeros of the function $J_1(\lambda c)$. Similarly it follows from equation (3.41) that this choice of biharmonic function automatically satisfies the boundary condition (3.1). The remaining boundary conditions (3.2) and (3.3) are satisfied provided that we can find constants a_n to satisfy the dual series equations

$$\sum_{n=1}^{\infty} \lambda_n^2 a_n J_0(\lambda_n \rho) = f(\rho), \qquad 0 \leq \rho < 1,$$

$$\sum_{n=1}^{\infty} \lambda_n a_n J_0(\lambda_n \rho) = 0, \qquad 1 < \rho \leq c$$

where the λ_i are the positive zeros of $J_1(\lambda c)$. We can readily transform these equations to the form considered by Sneddon and Srivastav (1963). If we multiply both sides of these equations by ρ and integrate the first with respect to ρ from 0 to ρ and the second from ρ to c we see that they are equivalent to the pair of dual series equations

$$\sum_{n=1}^{\infty} \lambda_n a_n J_1(\lambda_n \rho) = F(\rho), \qquad 0 \leq \rho < 1, \tag{3.42}$$

$$\sum_{n=1}^{\infty} a_n J_1(\lambda_n \rho) = 0, \qquad 1 < \rho \leqslant c, \tag{3.43}$$

where λ_1, λ_2, ..., λ_n, ... are the positive zeros of $J_1(\lambda c)$ and

$$F(\rho) = \frac{1}{\rho}\int_0^\rho uf(u)\,du. \tag{3.44}$$

It has been shown by Sneddon and Srivastav (1963) that the solution of these dual equations is

$$a_n = \frac{2}{c^2 J_0^2(\lambda_n c)}\int_0^1 u\,\alpha(u) J_1(u\lambda_n)du, \tag{3.45}$$

where

$$\alpha(\rho) = \sum a_n J_1(\lambda_n \rho) = -\frac{\partial}{\partial\rho}\int_\rho^1 \frac{g(t)\,dt}{\surd(t^2 - \rho^2)}, \quad 0 < \rho < 1, \tag{3.46}$$

and $g(t)$ is the solution of the integral equation (3.3) where $K(t, u)$ is the function defined by equations (3.10), (3.11) and

$$h(t) = \frac{2}{\pi}\, t^{-1}\frac{d}{dt}\int_0^t \frac{u^2 F(u)\,du}{\surd(t^2 - u^2)} = \frac{2}{\pi}\int_0^t \frac{u\,f(u)\,du}{\surd(t^2 - u^2)} \tag{3.47}$$

and so has the value given by equation (2.6). We therefore obtain the same result as before.

3.3. Solution of Problem (a) in the case of constant internal pressure.

We now consider the case in which the crack is opened up by a constant pressure p_0. The corresponding value of $h(t)$ is given by equation (3.16) so that if we make the substitution (3.17) for $g(t)$ we find that the relevant form of the integral equation (3.33) is

$$\phi(t) - \int_0^1 \phi(u)\,K(u, t)\,du = \frac{2}{\pi}\,t. \tag{3.48}$$

We begin by finding an iterative solution of this equation which is useful in discussing problems in which $c >> 1$.

If we write

$$\phi(t) = \frac{2}{\pi} \sum_{r=0}^{\infty} h_r(t) \tag{3.49}$$

then we have the iterative scheme

$$h_0(t) = t, \quad h_{r+1}(t) = \int_0^1 h_r(u)K(u, t)du, \quad (r = 0, 1, 2, \ldots) \tag{3.50}$$

for the determination of the functions $h_r(t)$.

We now expand the kernel $K(u, t)$ in powers of c^{-1}. Using the expansion

$$\cosh\left(\frac{u+t}{c}\right) v - \cosh\left(\frac{u-t}{c}\right) v$$

$$= \frac{2ut}{c^2} v^2 + \frac{(u^3t + ut^3)}{3c^4} v^4 + \frac{(3u^5t + 10u^3t^3 + 3ut^5)}{180\, c^6} v^6$$

$$+ \frac{(u^7 t + 7u^5 t^3 + 7u^3 t^5 + ut^7)}{2702\, c^8} v^8 + O(c^{-10}),$$

we see that we may express $K(u, t)$ in the form

$$K(u, t) = \sum_{r=0}^{\infty} \alpha_{2r+1}(u)t^{2r+1}, \tag{3.51}$$

where

$$\alpha_1(u) = \frac{2u}{\pi^2 c^3}\left\{T_2 + \frac{u^2 T_4}{3c^2} + \frac{u^4 T_6}{60c^4} + \frac{u^6 T_8}{2520c^6} + \cdots\right\},$$

$$\alpha_3(u) = \frac{2u}{3\pi^2 c^5}\left\{T_4 + \frac{u^2 T_6}{6c^2} + \frac{u^4 T_8}{120c^4} + \cdots\right\},$$

$$\alpha_5(u) = \frac{u}{30\pi^2 c^7}\left\{T_6 + \frac{u^2 T_8}{6c^2} + \dots\right\},$$

$$\alpha_7(u) = \frac{u}{1260\pi^2 c^9}\left\{T_8 + \dots\right\},$$

and we have written

$$T_i = \int_0^\infty \frac{K_1(x)}{I_1(x)} x^i \, dx. \tag{3.52}$$

The values of the integrals T_i are listed in Cooke & Tranter (1959).

From the form now taken by the function $K(u, t)$ it follows easily that the functions $h_r(t)$ may be written in the form

$$h_r(t) = \sum_{s=1}^{\infty} P^{(r)}_{2s+1} \, t^{2s+1}. \tag{3.53}$$

Suppose, for example, that c is sufficiently large to enable us to omit powers of t higher than the seventh. The next iteration $h_{r+1}(t)$ may then be obtained from equation (3.53) where the new coefficients will be determined from the equation

$$P_j^{(r+1)} = \int_0^1 \left\{P_1^{(r)} u + P_3^{(r)} u^3 + P_5^{(r)} u^5 + P_7^{(r)} u^7\right\} \alpha_i(u) du. \tag{3.54}$$

In the case of $h_0(t)$, the initial value, we take

$$P_1^{\,0} = 1, \quad P_i^{\,0} = 0, \quad i \neq 1. \tag{3.55}$$

Then, to find the coefficients of successive iterations we need only calculate the sixteen quantities

$$a_{s,\,r} = \int_0^1 u^s \alpha_r(u) \, du, \quad (r, s = 1,3,5,7). \tag{3.56}$$

The general term $h_r(t)$ will be given by (3.53) with $P_i^{(r)}$ satisfying the recurrence relation

$$P_i^{(r)} = a_{1,\,i} P_1^{(r-1)} + a_{3,\,i} P_3^{(r-1)} + a_{5,i} P_5^{(r-1)} + a_{7,i} P_7^{(r-1)}. \tag{3.57}$$

If we take

$T_2 = 2.50330, \quad T_4 = 3.77139, \quad T_6 = 23.431, \quad T_8 = 302.29$

we find that

$$a_{1,1} = \frac{2}{3\pi^2 c^3}\left\{2T_2 + \frac{T_4}{5c^2} + \frac{T_6}{140c^4} + \frac{T_8}{7560c^6}\right\} = \frac{0.3382}{c^3} + \frac{0.0509}{c^5} + \frac{0.0113}{c^7} + \frac{0.0026}{c^9},$$

$$a_{3,1} = \frac{2}{\pi^2 c^3}\left\{\frac{2T_2}{5} + \frac{T_4}{21c^2} + \frac{T_6}{540c^4} + \frac{T_8}{27720c^6}\right\} = \frac{0.2029}{c^3} + \frac{0.0364}{c^5} + \frac{0.0088}{c^7} + \frac{0.0022}{c^9},$$

$$a_{5,1} = \frac{2}{\pi^2 c^3}\left\{\frac{2T_2}{7} + \frac{T_4}{27c^2} + \frac{T_6}{660c^4} + \frac{T_8}{32760c^6}\right\} = \frac{0.1449}{c^3} + \frac{0.0283}{c^5} + \frac{0.0072}{c^7} + \frac{0.0019}{c^9},$$

$$a_{7,1} = \frac{2}{3\pi^2 c^3}\left\{\frac{2T_2}{3} + \frac{T_4}{11c^2} + \frac{3T_6}{260c^4} + \frac{T_8}{12600c^6}\right\} = \frac{0.1127}{c^3} + \frac{0.0232}{c^5} + \frac{0.0061}{c^7} + \frac{0.0016}{c^9},$$

$$a_{1,3} = \frac{2}{9\pi^2 c^5}\left\{T_4 + \frac{T_6}{10c^2} + \frac{T_8}{280c^4}\right\} = \frac{0.08492}{c^5} + \frac{0.0528}{c^7} + \frac{0.0243}{c^9},$$

$$a_{3,3} = \frac{2}{3\pi^2 c^5}\left\{\frac{T_4}{5} + \frac{T_6}{42c^2} + \frac{T_8}{1080c^4}\right\} = \frac{0.0510}{c^5} + \frac{0.0377}{c^7} + \frac{0.0190}{c^9},$$

$$a_{5,3} = \frac{2}{3\pi^2 c^5}\left\{\frac{T_4}{7} + \frac{T_6}{54c^2} + \frac{T_8}{1320c^4}\right\} = \frac{0.0364}{c^5} + \frac{0.0293}{c^7} + \frac{0.0155}{c^9} ,$$

$$a_{7,3} = \frac{2}{9\pi^2 c^5}\left\{\frac{T_4}{3} + \frac{T_6}{22c^2} + \frac{T_8}{520c^4}\right\} = \frac{0.0283}{c^5} + \frac{0.0240}{c^7} + \frac{0.0131}{c^9} ,$$

$$a_{1,5} = \frac{1}{90\pi^2 c^7} \quad T_6 + \frac{T_8}{10c^2} = \frac{0.0264}{c^7} + \frac{0.0340}{c^9} ,$$

$$a_{3,5} = \frac{1}{30\pi^2 c^7}\left\{\frac{T_6}{5} + \frac{T_8}{42c^2}\right\} = \frac{0.0158}{c^7} + \frac{0.0243}{c^9} ,$$

$$a_{5,5} = \frac{1}{30\pi^2 c^7}\left\{\frac{T_6}{7} + \frac{T_8}{54c^2}\right\} = \frac{0.0113}{c^7} + \frac{0.0189}{c^9} ,$$

$$a_{7,5} = \frac{1}{90\pi^2 c^7}\left\{\frac{T_6}{3} + \frac{T_8}{22c^2}\right\} = \frac{0.0088}{c^7} + \frac{0.01547}{c^9} ,$$

$$a_{1,7} = \frac{T_8}{3780\pi^2 c^9} = \frac{0.0081}{c^9} ,$$

$$a_{3,7} = \frac{T_8}{6300\pi^2 c^9} = \frac{0.0049}{c^9} ,$$

$$a_{5,7} = \frac{T_8}{8820\pi^2 c^9} = \frac{0.0035}{c^9} ,$$

$$a_{7,7} = \frac{T_8}{11340\pi^2 c^9} = \frac{0.0027}{c^9} .$$

For instance, it is easily shown that up to terms of order c^{-7},

$$P_1^{(0)} = 1,\ P_1^{(1)} = a_{1,1},\ P_3^{(1)} = a_{1,3},\ P_5^{(1)} = a_{1,5},\ P_1^{(2)} = a_{1,1}\, a_{1,1} +$$

$$+ a_{3,1}\, a_{1,3} + a_{5,1}\, a_{1,5} = \frac{0.1144}{c^6}$$

and the others may be neglected. Since

$$\phi(t) = \frac{2}{\pi} \sum_{r,\, s} P_{2s+1}^{(r)}\, t^{2s+1}$$

it follows from equation (3.19) that

$$\omega_1(c^{-1}) = 3 \sum_{r,\, s} \frac{P_{2s+1}^{(r)}}{2s+3}$$

and we can easily show that

$$\omega_1(x) = 1 + 0.3382\, x^3 + 0.0815\, x^5 + 0.1144\, x^6 + 0.0125\, x^7 + 0(x^8).$$

From the definition (3.22) we then have

$$\Omega_1(x) = 1 - 0.3382\, x^3 - 0.1087\, x^5 - 0.1856\, x^6 - 0.0219\, x^7 + 0(x^8)$$

showing that, in the notation of equation (3.23), the critical value of the pressure for a crack radius $a(\ll c)$ is given by the equation

$$p_{cr} = p_{cr}^{(\infty)}\Big\{1 - 0.3382a^3/c^3 - 0.1087a^5/c^5 - 0.1856a^6/c^6 - $$
$$- 0.0219a^7/c^7 + O(a^8/c^8)\Big\}. \qquad (3.58)$$

If we define a ratio Π_1 by the equation

$$\Pi_1 = \frac{p_{cr}^{(\infty)} - p_{cr}}{p_{cr}^{\infty}} \times 100 \qquad (3.59)$$

so that Π_1 denotes the percentage change in the value of the critical pressure p_{cr} from the value $p_{cr}^{(\infty)}$ corresponding to a crack in an infinite solid, then if $c \gg a$,

$$\Pi_1 = 33.82a^3/c^3 + 10.87a^5/c^5 + 18.56a^6/c^6 + 2.19a^7/c^7 + O(a^8/c^8). \qquad (3.59a)$$

These formulae are of use only if the ratio a/c is small. When it is only slightly less than unity the integral equation (3.48) has to be solved numerically. At first sight the solution $\phi(t)$ of this equation would appear to depend only on t in the range $0 \leq t \leq 1$ but it will be recalled that the kernel $K(u, t)$, defined by equations (3.35) and (3.36), depends on the value of <u>c</u>, the radius of the cylinder expressed in terms of the radius of the crack as the unit of length. The computations were carried out for the values c = 1.05, 1.10, 1.20, 1.30, 1.6667. The values of the integral $H(\lambda)$ defined by equation (3.36) were first calculated using Weddle's rule for numerical integration and the integral equation itself was solved using the method of Fox and Goodwin (195). The calculations were carried out on the DEUCE computer in the Computing Laboratory of the University of Glasgow.

The results of the calculations of $\phi(t)$ are shown in Table 3 and graphically in Fig. 7. It will be observed that even for moderately small values of <u>c</u> (for instance greater than 1.20) the graph of the function $\phi(t)$ differs very little from a straight line.

Using these values of $\phi(t)$ and the formulae (3.18) and (3.19) we

Table 3. The variation with c and t of $\phi(t)$.

12 t \ c	1.05	1.10	1.20	1.30	1.6667
1	.0893	.0785	.0690	.0639	.0575
2	.1791	.1574	.1382	.1285	.1151
3	.2702	.2370	.2077	.1926	.1727
4	.3632	.3177	.2777	.2579	.2304
5	.4591	.3998	.3486	.3227	.2881
6	.5591	.4833	.4204	.3890	.3460
7	.6649	.5726	.4937	.4551	.4039
8	.7794	.6639	.5688	.5231	.4621
9	.9068	.7641	.6463	.5916	.5205
10	1.0556	.8719	.7272	.6620	.5791
11	1.2445	.9965	.8126	.7341	.6380
12	1.5304	1.1479	.9047	.8092	.6972

Table 4. The variation of W/W_∞, Π_1 and n_1 with c/a.

c/a	1.05	1.10	1.20	1.30	1.6667	2.5	5.0
W/W_∞	1.9704	1.6649	1.3670	1.2725	1.0994	1.0205	1.0027
Π_1	33.3	26.9	16.3	12.5	6.9	2.1	0.3
n_1	140	80	42	27	9.5	2.3	0.3

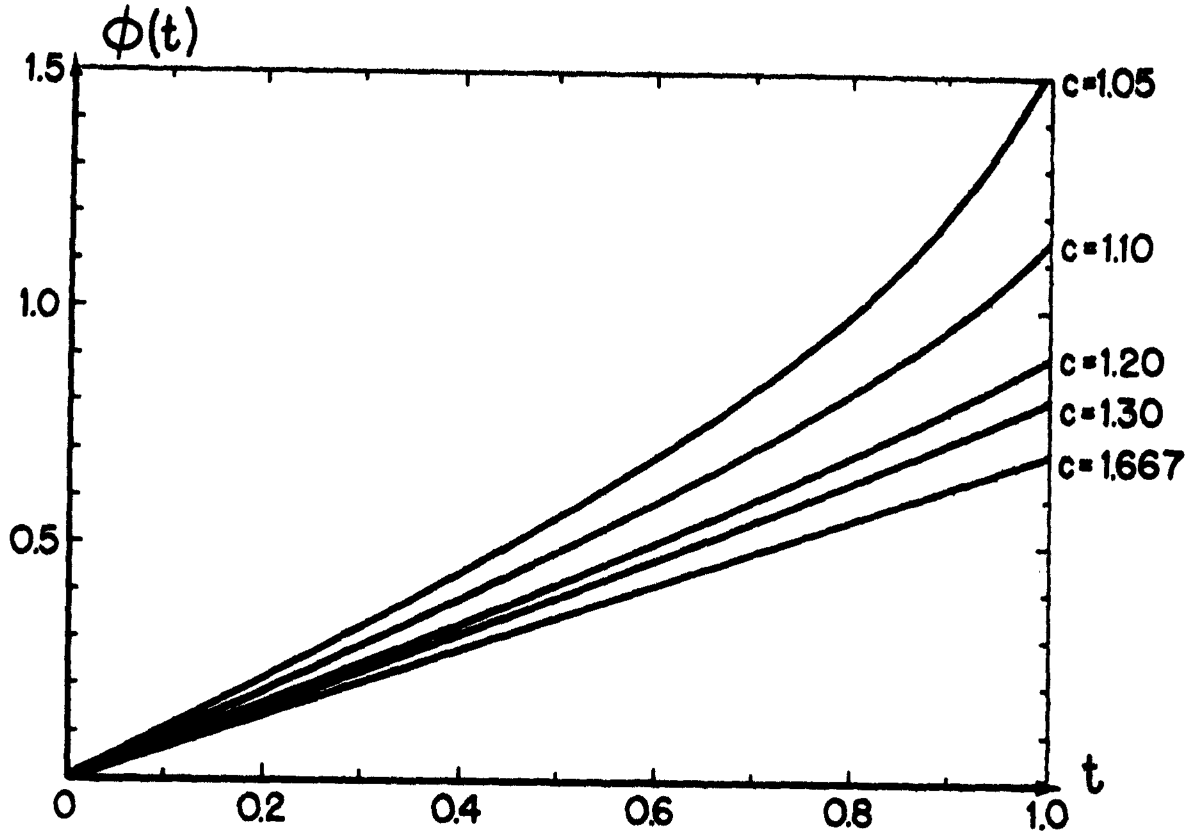

Figure 7

Fig. 7 The variation with t and c of the function $\Phi(t)$. (a is taken to be unity).

easily calculate the variation with a/c of the ratio W/W_∞ where W_∞ denotes the value of W corresponding to $c = \infty$, i.e. to a value 0 of the ratio a/c, so that

$$W_\infty = \frac{8\, p_o^2 (1 - \eta^2) a^3}{3\, E} .$$

The results of these calculations are shown in Table 4 and graphically in Fig. 8.

To illustrate the variation of the critical pressure, p_{cr}, necessary to cause the spread of the crack when its radius is a the ratio Π_1 defined by equations (3.59), (3.23), (3.22) and (3.19) was calculated. The results are shown in Table 4 and graphically in Fig. 9. From this diagram it follows that the drop in the value of the critical pressure $p_{cr}^{(\infty)}$ from the value p_{cr} corresponding to a crack in an infinite solid is less than 5% if $c > 2a$ and is less than 10% if $c > 1.4a$.

If we look at the variation with a/c of the stress intensity factor N we get much the same kind of picture. To illustrate the variation of N we calculated n_1 the percentage increase in the stress intensity factor due to the effect of the finite radius of the cylinder. The results are shown in the last row of Table 4 and graphically in Fig. 10. The effect of the finite value of the radius is more pronounced in this case. For instance if $c < 1.2a$, the change in the stress intensity factor is greater than 40%. However it is less than 5% if $c > 2a$ and is less than 10% if $c > 1.7a$.

It is of interest to compare these results with those obtained in the analogous problem in plane strain. If we plot the ratio Π_1 as a function of x, the ratio of the area of the crack to the cross-sectional area of the cylinder (i.e. $x = a^2/c^2$) we get the curve shown in Fig. 11. The curve Π shown in the same diagram shows the value of the percentage change in p_{cr} in the plane strain problem but here x is taken to be the ratio of the length of the crack to the width of the strip. In Fig. 12 we show a comparison of n_1 the percentage increase in the stress intensity factor in the axisymmetric case with n the corresponding quantity in the plane strain case, x being given the same interpretation as before. Using this basis of comparison we see that as far as an "engineering" approximation is concerned the size effect in the axisymmetric case can be gauged from that in the plane strain case.

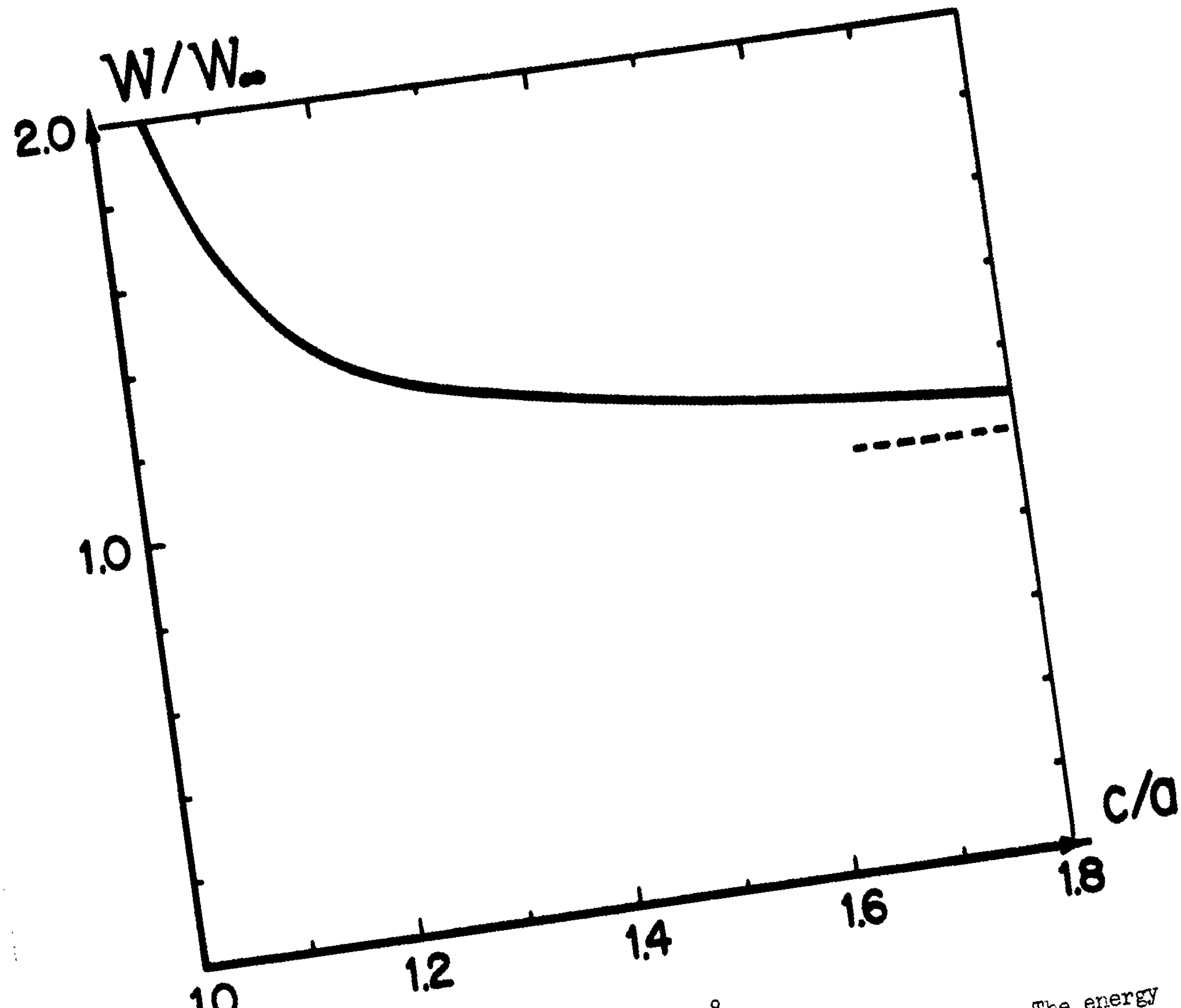

Figure 8

Fig. 8 The variation with c/a of the energy W_1 of the crack. The energy W_∞ is $8p_o^{\,2}(1-\eta^2)a^3/3E$.

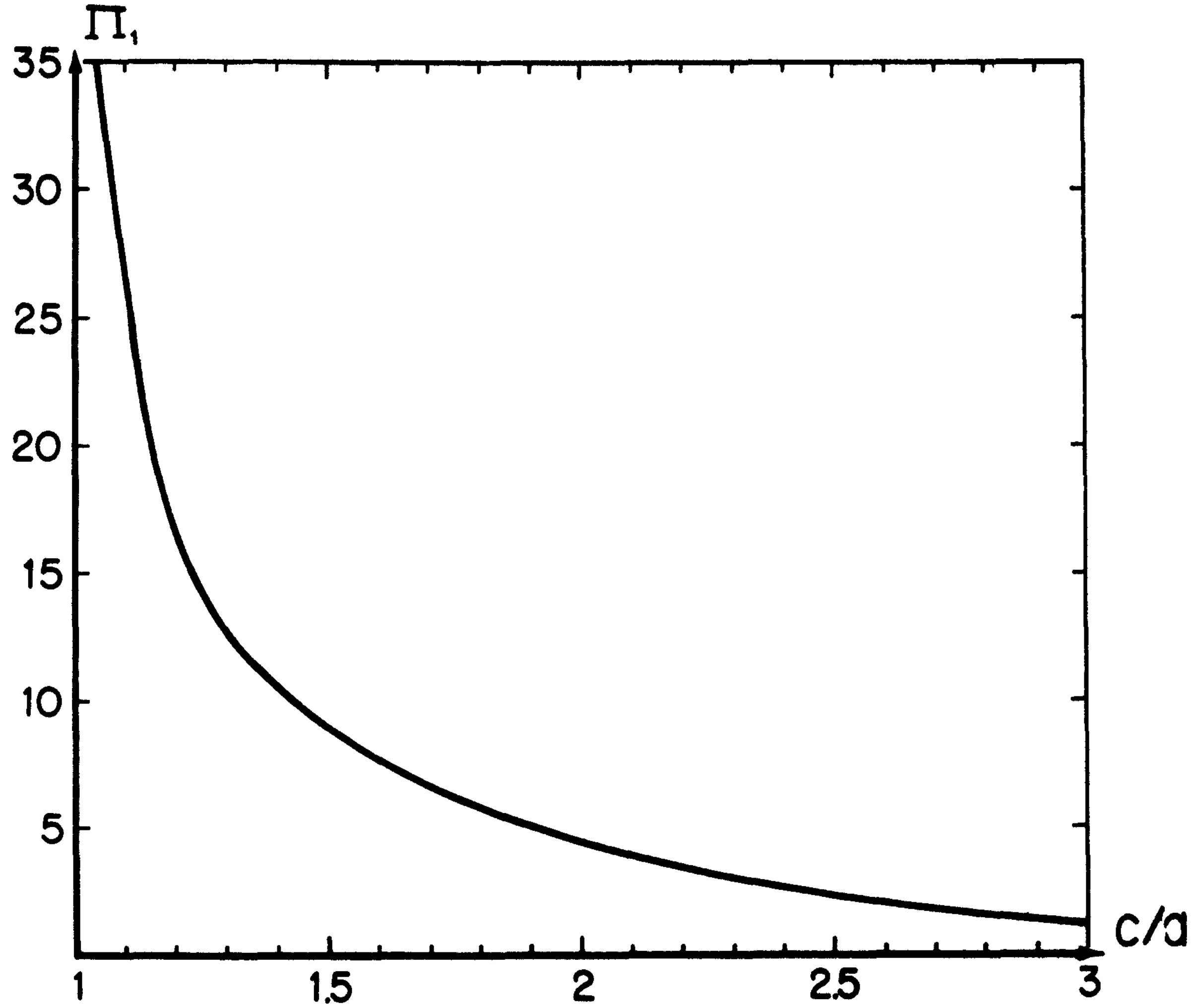

Figure 9

Fig. 9 The variation with c/a of π_1, the percentage change in the value of the critical pressure p_{cr} from the value $p_{cr}^{(\infty)}$ corresponding to a crack in an infinite solid.

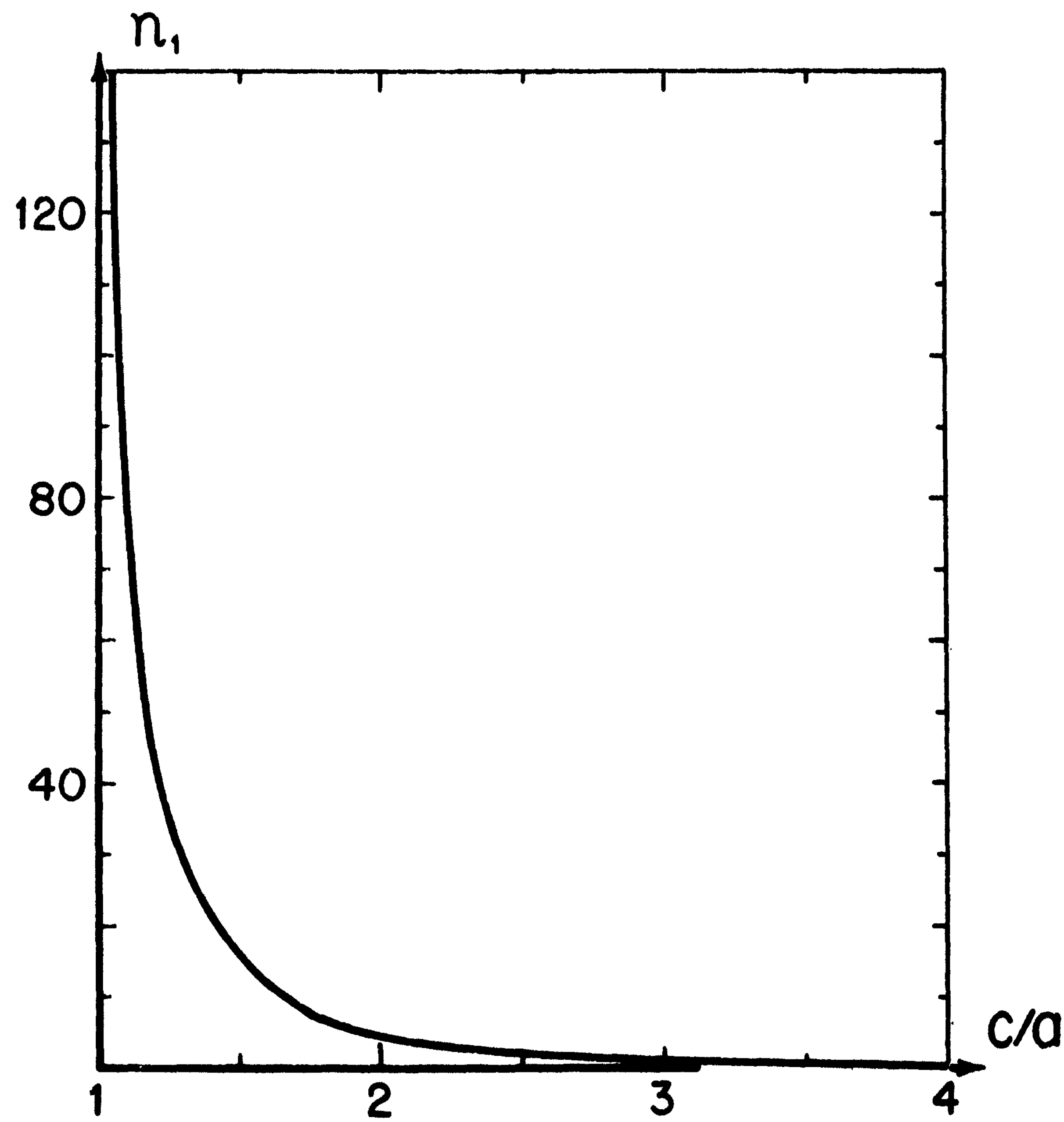

Figure 10

Fig. 10 The variation with c/a of n_1 the percentage increase in the stress intensity factor due to the effect of finite radius of the cylinder.

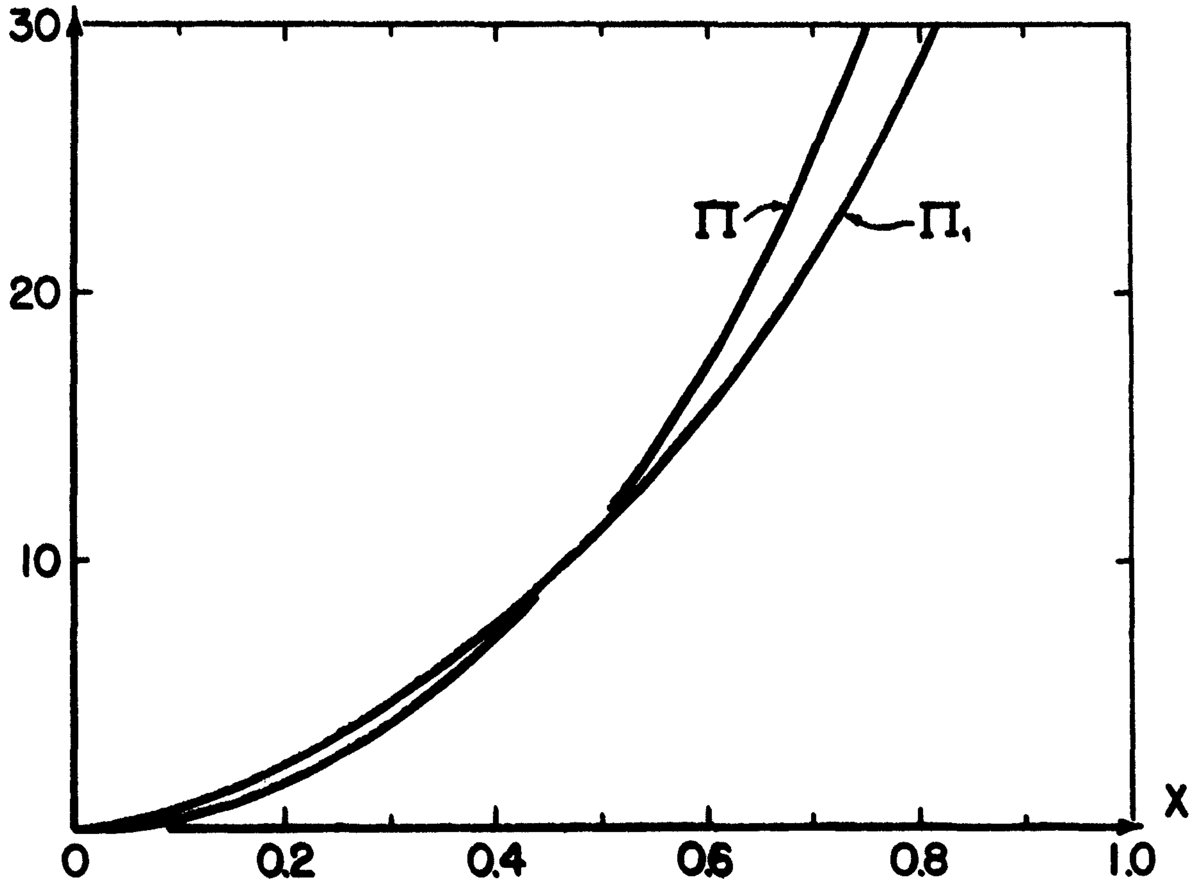

Figure 11

Fig. 11 A comparison of the axisymmetric solution and the plane strain solution. π_1 is the percentage change in the value of the critical pressure in the axisymmetric case with x equal to the ratio of the area of the crack to the cross-section area of the cylinder; π is the analogous quantity in the plane strain problem with x equal to the ratio of the length of the crack to the width of the strip.

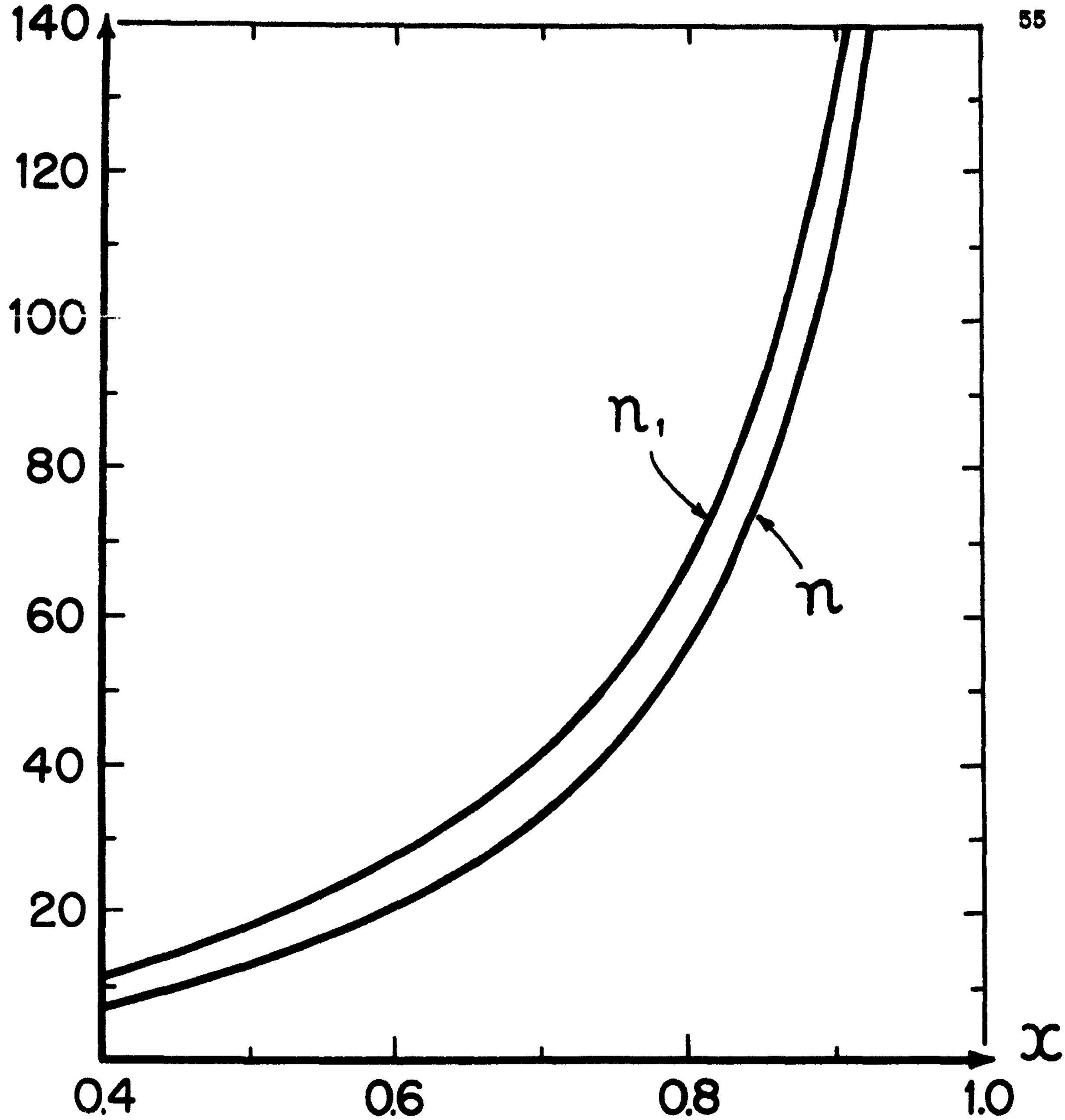

Figure 12

Fig. 12 A comparison of the axisymmetric solution and the plane strain solution. n_1 is the percentage increase in the stress intensity factor in the axisymmetric case, $\underline{n}$ that in the plane strain case. ($\underline{x}$ is defined as in Fig. 11).

3.4. Solution of Problem (b).

In the second of the two problems we shall consider the conditions on the crack surface are the same as in the first problem but the conditions on the curved surface of the cylinder are different. Instead of assuming that the normal component of the displacement vanishes we assume that the normal component of the stress vanishes, i.e. the boundary conditions (3.4) are replaced by the conditions

$$\sigma_{\rho\rho} = \sigma_{\rho z} = 0, \qquad \rho = c, \quad 0 \leq z < \infty \tag{3.60}$$

If we take the form (3.6) for the stress-function $\chi(\rho, z)$ we find that on the surface $\rho = c$ the stress component $\sigma_{\rho\rho}$ assumes the form

$$\begin{aligned}\Big[\sigma_{\rho\rho}\Big]_{\rho=c} = \frac{2\mu}{c}\int_0^\infty \Big\{A(\xi)\Big[c\xi I_0(c\xi) - I_1(c\xi)\Big] + B(\xi)\Big[(3-2\eta)c\xi I_0(c\xi) - \\ - 4(1-\eta)I_1(c\xi) - c^2\xi^2 I_1(c\xi)\Big]\Big\}\cos(\xi z)\,d\xi \\ - 2\mu\int_0^\infty F(\zeta)e^{-\zeta z}\Big\{J_0(\zeta c) - (1-2\eta)\frac{J_1(\zeta c)}{\zeta c} - \zeta z \times \\ \Big[J_0(\zeta c) - \frac{J_1(\zeta c)}{\zeta c}\Big]\Big\}d\zeta \ .\end{aligned}$$

If we take the Fourier cosine transform of both sides of this equation and make use of the fact that $\big[\sigma_{\rho\rho}\big]_{\rho=c} = 0$ and hence that $\mathcal{F}_c\Big\{\Big[\sigma_{\rho\rho}\Big]_{\rho=c}\,;\ z \to \xi\Big\}$ we find that

$$\begin{aligned}A(\xi)\Big[c\xi I_0(c\xi) - I_1(c\xi)\Big] + B(\xi)\Big[(3-2\eta)c\xi I_0(c\xi) - 4(1-\eta)\times \\ I_1(c\xi) - c^2\xi^2 I_1(c\xi)\Big] = c\xi i_3 + \eta i_1 - i_2\end{aligned} \tag{3.61}$$

where i_1 and i_2 are defined by equations (3.28) and i_3 is defined by the equation

$$i_3 = \frac{4\xi}{\pi} \quad \frac{\zeta F(\zeta)\, J_0(\zeta c)\, d\zeta}{(\xi^2 + \zeta^2)^2} \ . \tag{3.62}$$

Substituting the expression (3.9) for F( ) into equation (3.62), interchanging the order of the integrations and making use of well-known integrals involving Bessel functions (Cf. pp.10, 11 of vol.ii of Erdelyi (1954)), we find that

$$i_3 = \frac{2}{\pi}\int_0^1 g(u)\left\{\sinh(\xi u)\left[K_0(c\xi) - c\xi K_1(c\xi)\right] + \xi u\cosh(\xi u)\times K_0(c\xi)\right\}du. \tag{3.63}$$

Solving the equations (3.61), (3.27) we find that the boundary conditions (3.60) are satisfied if we choose $A(\xi)$, $B(\xi)$ to be given by the expressions

$$\tfrac{1}{2}\pi A(\xi)G(c\xi) = \left[H(c\xi) - 2 + 2\eta\right]\int_0^1 g(u)\,\xi u\cosh(\xi u)\,du + \left[(3-2\eta)H(c\xi) - 4 + 4\eta - c^2\xi^2\right]\int_0^1 g(u)\sinh(\xi u)\,du, \tag{3.64}$$

$$\tfrac{1}{2}\pi B(\xi)G(c\xi) = \int_0^1 g(u)\,\xi u\cosh(\xi u)\,du + \left[1 - H(c\xi)\right]\int_0^1 g(u)\sinh(\xi u)\,du \tag{3.65}$$

where the function G is defined by the equation

$$G(s) = s^2 I_0^2(s) - (2 - 2\eta + s^2)I_1^2(s) \tag{3.66}$$

and the function H by the equation

$$H(s) = s^2 I_0(s)K_0(s) + (2 - 2\eta + s^2)I_1(s)K_1(s). \tag{3.67}$$

If we substitute from equations (3.64) and (3.65) into equation (3.11) we find that this last equation reduces to the integral equation

$$g(t) + \frac{2}{\pi}\int_0^1 L(t,u)g(u)\,du = h(t), \quad 0 \leqslant t \leqslant 1, \tag{3.68}$$

for the determination of the function $g(t)$, the kernel $L(t, u)$ being defined by the equation

$$L(t, u) = \frac{2}{\pi}\int_0^\infty \frac{1}{G(c\xi)}\Big\{\Big[H(c\xi) - 1\Big]\Big[\xi u \cosh(\xi u)\sinh(\xi t) + \xi t \cosh(\xi t) \times$$

$$\sinh(\xi u)\Big] + \Big[2H(c\xi) - 3 + 2\eta - c^2\xi^2\Big]\sinh(\xi u)\sinh(\xi t)$$

$$- \xi^2\, u\, t \cosh(\xi u)\cosh(\xi t)\Big\}\, d\xi. \qquad (3.69)$$

The equations (3.68) and (3.69) are identical with those derived by Collins [Cf. equations (3.13) and (3.14) of Collins (1962)].

For small values of the constant c^{-1} it is possible to solve the integral equation (3.68) by an iterative procedure. To derive this solution we need first to derive an expansion of $L(t, u)$ in powers of c^{-1}. We find that

$$L(t, u) = \sum_{r=0}^{\infty} c^{-2r-3} \sum_{n=0}^{r} D_{r-n,n}\, t^{2n+1}\, u^{2r-2n+1} \qquad (3.70)$$

where

$$D_{m,n} = D_{n,m} = 2\Big[(m + n + 2)E_{m,n} - \Big\{2(m + 1)(n + 1) + 1 - \eta\Big\}F_{m,n} -$$

$$- (m + 1)(2m + 3)F_{m+1,n}\Big] \qquad (3.71)$$

with

$$E_{m,n} = E_{n,m} = \frac{2}{\pi(2m + 1)!(2n + 1)!}\int_0^\infty \frac{s^{2m+2n+2}H(s)\,ds}{G(s)}, \qquad (3.72)$$

and

$$F_{m,n} = F_{n,m} = \frac{2}{\pi(2m + 1)!(2n + 1)!}\int_0^\infty \frac{s^{2m+2n+2}\,ds}{G(s)}. \qquad (3.73)$$

For a prescribed value of the Poisson ratio η the coefficients $E_{m,n}$ and $F_{m,n}$ have to be evaluated numerically for various values of m and n depending on the order of the terms retained in the expansion on the right-hand side of the equation (3.70).

3.5. Solution of Problem (b) in the Case of Constant Internal Pressure.

We now consider in more detail the case in which the internal pressure in the crack is p_0 so that $f(\rho) = p_0/2\mu$ and $h(t) = p_0 t/(\pi\mu)$ and the integral equation (3.68) becomes

$$g(t) + \frac{2}{\pi}\int_0^1 L(t, u)g(u)du = \frac{p_0 t}{\pi\mu}, \quad 0 \leq t \leq 1. \tag{3.74}$$

Collins has obtained the approximate solution

$$\begin{aligned} g(t) = \frac{p_0 t}{\pi\mu}\Bigg[1 &- \frac{2D_{00}}{3\pi c^3} - \frac{2D_{10}}{3\pi c^5}\left(t^2 + \frac{3}{5}\right) + \frac{4D_{00}^2}{9\pi^2 c^6} \\ &- \frac{2}{\pi c^7}\left(\frac{D_{20}t^4}{3} + \frac{D_{11}t^2}{5} + \frac{D_{20}}{7}\right) + \frac{4D_{00}D_{10}}{9\pi^2 c^8}\left(t^2 + \frac{9}{5}\right) \\ &- \frac{2}{\pi c^9}\left(\frac{D_{30}t^6}{3} + \frac{D_{21}t^4}{5} + \frac{D_{21}t^2}{7} + \frac{D_{30}}{9} + \frac{4D_{00}^3}{27\pi^2}\right) \\ &+ \frac{4}{\pi^2 c^{10}}\Bigg\{\frac{D_{00}D_{20}t^4}{9} + (D_{00}D_{11} + 2D_{10}^2)\frac{t^2}{15} + \\ &+ \left(\frac{D_{00}D_{20}}{7} + \frac{46D_{10}^2}{525} + \frac{D_{00}D_{11}}{25}\right)\Bigg\}\Bigg] + O(c^{-11}) \end{aligned} \tag{3.75}$$

by the method of iteration.

Now by equations (3.13), (3.14) we find that the energy of the crack is given by the equation

$$W_2 = 4\pi p_0(1 - \eta)\int_0^1 tg(t)dt \tag{3.76}$$

so that in our units (in which the radius of the crack is taken to be the unit of length)

$$\begin{aligned} W_2 = \frac{4(1-\eta)p_0^2}{3\mu}\Bigg\{1 &- \frac{2D_{00}}{3\pi c^3} - \frac{4D_{10}}{5\pi c^5} + \frac{4D_{00}^2}{9\pi^2 c^6} - \frac{2}{\pi c^7}\left(\frac{2D_{20}}{7} + \frac{3D_{11}}{25}\right) \\ &+ \frac{16D_{00}D_{10}}{15\pi^2 c^8} - \frac{4}{\pi c^9}\left(\frac{D_{30}}{9} + \frac{3D_{21}}{35} + \frac{2D_{00}}{27\pi^2}\right) + \frac{8}{\pi^2 c^{10}}\left(\frac{2D_{00}D_{20}}{21} + \frac{D_{00}D_{11}}{25}\right) + \\ &+ \frac{44D_{10}^2}{525} + O(c^{-11})\Bigg\}. \end{aligned}$$

In conventional units this becomes

$$W_2 = \frac{8(1-\eta^2)p_0^2 a^3}{3E}\left\{1 - \frac{2D_{00}a^3}{3\pi c^3} - \frac{4D_{10}a^5}{5\pi c^5} + \frac{4D_{00}^2 a^6}{9\pi^2 c^6} - \frac{2a^7}{\pi c^7}\left(\frac{2D_{20}}{7} + \frac{3D_{11}}{25}\right)\right.$$

$$+ \frac{16 D_{00}D_{10}a^8}{15\pi^2 c^8} - \frac{4a^9}{\pi c^9}\left(\frac{D_{30}}{9} + \frac{3D_{21}}{35} + \frac{2D_{00}^3}{27\;{}^2}\right) + \frac{8a^{10}}{{}^2\,c^{10}}\left(\frac{2D_{00}D_{20}}{21} + \frac{D_{00}D_{11}}{25} + \frac{44D_{10}^2}{525}\right)$$

$$\left. + O(a^{11}/c^{11})\right\}. \tag{3.77}$$

Applying the Griffith criterion (3.20) we find that the critical value of p_0 is given by the formula

$$p_{cr} = p_{cr}^{(\infty)}\left[1 + \frac{2D_{00}a^3}{3\pi c^3} + \frac{16D_{10}a^5}{15\pi c^5} + \frac{2a^7}{\pi c^7}\left(\frac{10D_{20}}{21} + \frac{D_{11}}{5}\right) + \frac{8D_{00}D_{10}a^8}{45\pi^2 c^8}\right.$$

$$\left. + \frac{8a^9}{\pi c^9}\left(\frac{D_{30}}{9} + \frac{3D_{21}}{35}\right) + \frac{8a^{10}}{3\pi^2 c^{10}}\left(\frac{2D_{00}D_{21}}{21} + \frac{D_{00}D_{11}}{25} + \frac{2D_{10}^2}{21}\right) + O(a^{11}/c^{11})\right] \tag{3.78}$$

where, as before, $p_{cr}^{(\infty)} = \left\{\pi ET/2(1-\eta^2)a\right\}^{\frac{1}{2}}$.

In the case $\eta = 0.3$ we find that this formula takes the form

$$p_{cr} = p_{cr}^{(\infty)}\left\{1 - 0.2626\,a^3/c^3 - 0.0340\,a^5/c^5 - 0.4570\,a^7/c^7 + 0.0176\,a^8/c^8\right.$$

$$\left. + O(a^9/c^9)\right\} \tag{3.79}$$

If we define a percentage change Π_2 in the value of the critical pressure p_{cr} from the value $p_{cr}^{(\infty)}$ in the infinite case by equation (3.59) then we find that

$$\Pi_2 = 26.26\,a^3/c^3 + 3.40\,a^5/c^5 + 45.70\,a^7/c^7 + O(a^8/c^8). \tag{3.80}$$

Formulae (3.79) and (3.80) should be compared with their counterparts (3.58) and (3.59) derived for the boundary conditions of Problem (a).

When a/c is not very small we again have recourse to numerical integration of the integral equation. To facilitate the numerical calculations we transform the integral equation (3.68) by making the substitutions

$$g(t) = \frac{p_0 c}{\pi\mu}\,\phi(t/c), \quad K(t,\,u) = \frac{2c}{\pi}\,L(ct,\,cu), \qquad \alpha = c^{-1} \tag{3.81}$$

when it reduces in the case $h(t) = p_0 t/(\pi\mu)$ to the form

$$\phi(t) + \int_0^{\alpha} K(t,\,u)\,\phi(u)\,du = t, \qquad 0 \leqslant \alpha < 1 \tag{3.82}$$

where the kernel is now given by the equation

$$K(u,\,t) = \frac{4}{\pi^2}\int_0^{\infty}\Big\{\Big[2H(x) - 3 + 2\eta - x^2\Big]\sinh(u\,x)\sinh(t\,x) -$$

$$- u\,t\,x^2\cosh(u\,x)\cosh(t\,x) + x\Big[H(x) - 1\Big]\Big[u\cosh(u\,x)\sinh(t\,x) +$$

$$+ t\sinh(u\,x)\cosh(t\,x)\Big]\Big\}\frac{dx}{G(x)} \tag{3.83}$$

with $H(x)$ and $G(x)$ defined by equations (3.66) and (3.67).

From equations (3.13) and (3.14) we find that the energy W_2 required to open the crack is given (in conventional units) by the equation

$$W_2 = \frac{8\,p_0^2(1 - \eta^2)}{E}\,c^3\int_0^{a/c} t\,\phi(t)\,dt \tag{3.84}$$

where <u>a</u> is the radius of the crack, so that the Griffith criterion (3.20) that the crack may spread leads to the formula

$$p_{cr} = p_{cr}^{(\infty)}\;\Omega_2(a/c), \tag{3.85}$$

with

$$\Omega_2(\alpha) = \Big\{\alpha/\phi(\alpha)\Big\}^{\frac{1}{2}} \tag{3.86}$$

for the critical value of the applied pressure, $p_{cr}^{(\infty)}$ being defined as before.

The percentage change Π_2 in the value of $p_{cr}^{(\infty)}$ is therefore given by the formula

$$\Pi_2 = 100\,\Big\{1 - \Omega_2(a/c)\Big\} \tag{3.87}$$

In a similar way equation (3.12) leads to the expression

$$u_3^{\circ}(\rho) = \frac{4(1-\eta^2)p_0 a}{\pi E} F_\alpha(x), \qquad 0 \leqslant x < \alpha \tag{3.88}$$

with

$$\alpha = a/c, \quad x = \rho/c, \quad F_\alpha(x) = \frac{1}{\alpha}\int_x^\alpha \frac{\phi(t)dt}{\sqrt{(t^2 - x^2)}} \tag{3.89}$$

for the normal displacement of the surface of the crack. If we write

$$b = \frac{4(1-\eta^2)p_0 a}{\pi E} \tag{3.90}$$

which is the value of u_3 at the origin of coordinates in the case $c \gg a$ we see that we can write equation (3.88) in the form

$$u_3^{\circ}(\rho) = b\,F_\alpha(x). \tag{3.91}$$

The percentage increase n_2 in the stress intensity factor can be calculated by means of equation (3.60). We find that

$$n_2 = \left\{\frac{\phi(\alpha)}{\alpha} - 1\right\} \times 100, \quad \alpha = a/c. \tag{3.92}$$

The various functions were computed for the value $\eta = 0.25$.

The variation of $\phi(t)$ with $\underline{t}$ for six values of α is shown in Table 5 and graphically in Fig. 13, while that of $F_\alpha(x)$ for $0 \leqslant x < 1$ and the same values of α is shown in Table 6 and Fig. 14. The variation of the function $\Omega_2(a/c)$ i.e. of the ratio $p_{cr}/p_{cr}^{(\infty)}$ is given by the second row of Table 7. Fig. 15 shows the variation of the critical pressure p_{cr} with the ratio a/c. The values of the other quantities of physical interest Π_2 and n_2 are also listed in Table 7.

To illustrate the differing effects of the two kinds of boundary conditions on the cylindrical surface - those considered here and those considered in problem (a) - the variation of Π_1 and Π_2 is shown in Fig. 16 and that of n_1 and n_2 is shown in Fig. 17.

Using either of these quantities as a measure of the effect of the finite radius of the cylinder we see that if $a > \frac{1}{2}c$ the effect is more

Table 5: Values of the function $\phi(r\alpha/12)$ $(\eta = 0.25)$.

$r \backslash \alpha$	0.2	0.3	0.5	0.7	0.8	0.9
1				0.072	0.093	0.128
2	0.033	0.051	0.089	0.144	0.186	0.256
3				0.216	0.280	0.385
4	0.067	0.101	0.179	0.288	0.375	0.519
5				0.360	0.468	0.651
6	0.100	0.152	0.267	0.434	0.566	0.795
7				0.504	0.660	0.932
8	0.134	0.203	0.358	0.576	0.758	1.031
9				0.647	0.853	1.203
10	0.167	0.253	0.443	0.721	0.957	1.356
11				0.793	1.054	1.577
12	0.201	0.304	0.536	0.881	1.183	1.802

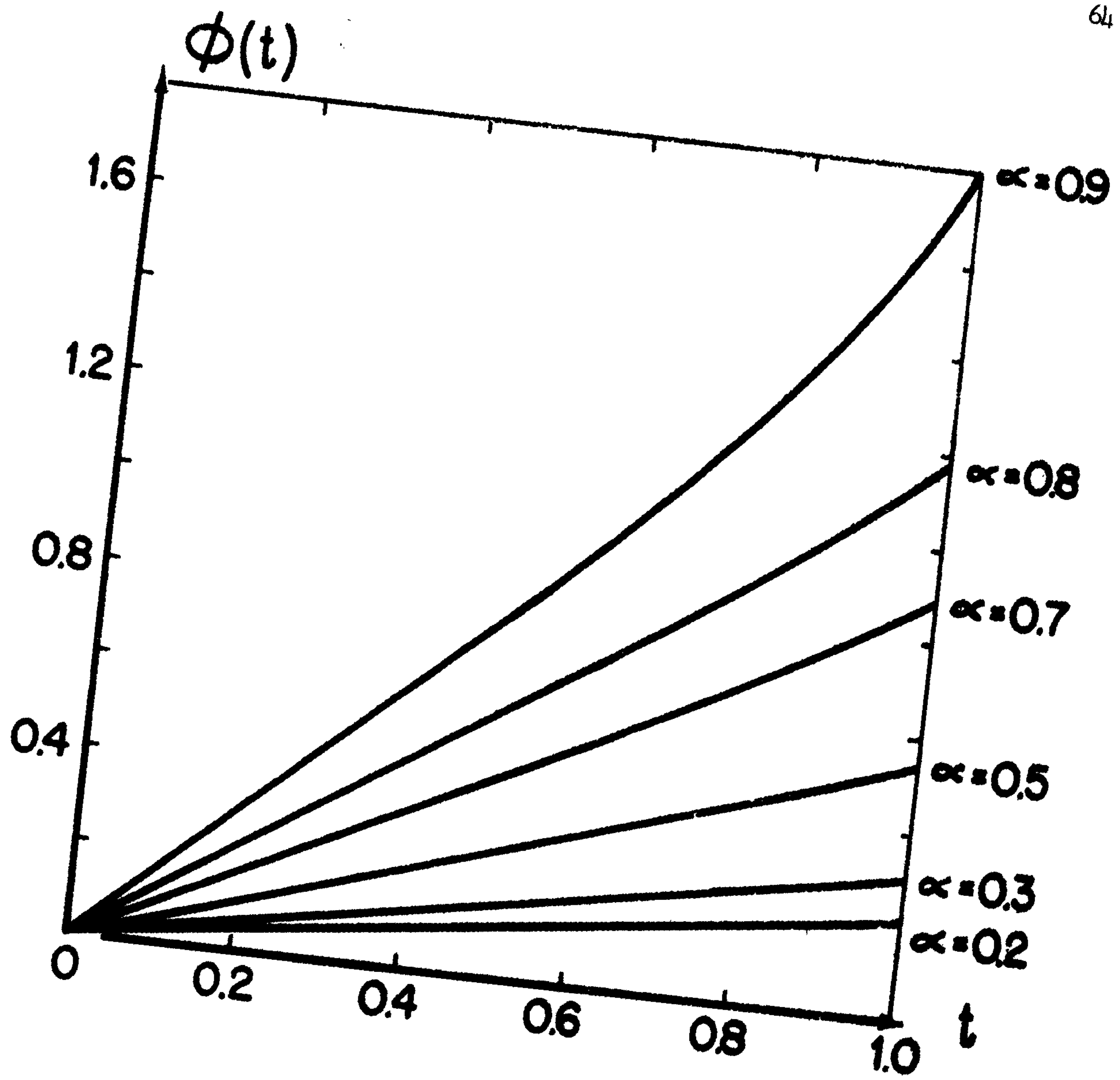

Figure 13

Fig. 13 The variation with t and α of the function $\Phi(t)$ for constant pressure p_o and $\eta = 0.25$.

Table 6: Values of the function $F_\alpha(r\,\alpha/12)$. (η = 0.25).

$r \backslash \alpha$	0.2	0.3	0.5	0.7	0.8	0.9
0	1.004	1.014	1.070	1.238	1.418	1.789
1				1.234	1.414	1.784
2	0.990	1.000	1.055	1.221	1.400	1.768
3				1.199	1.376	1.742
4	0.947	0.956	1.008	1.168	1.342	1.704
5				1.127	1.297	1.654
6	0.869	0.878	0.926	1.074	1.239	1.587
7				1.009	1.166	1.504
8	0.748	0.755	0.795	0.926	1.073	1.394
9				0.822	0.955	1.254
10	0.554	0.560	0.589	0.688	0.802	1.064
11				0.496	0.579	0.774
12	0.000	0.000	0.000	0.000	0.000	0.000

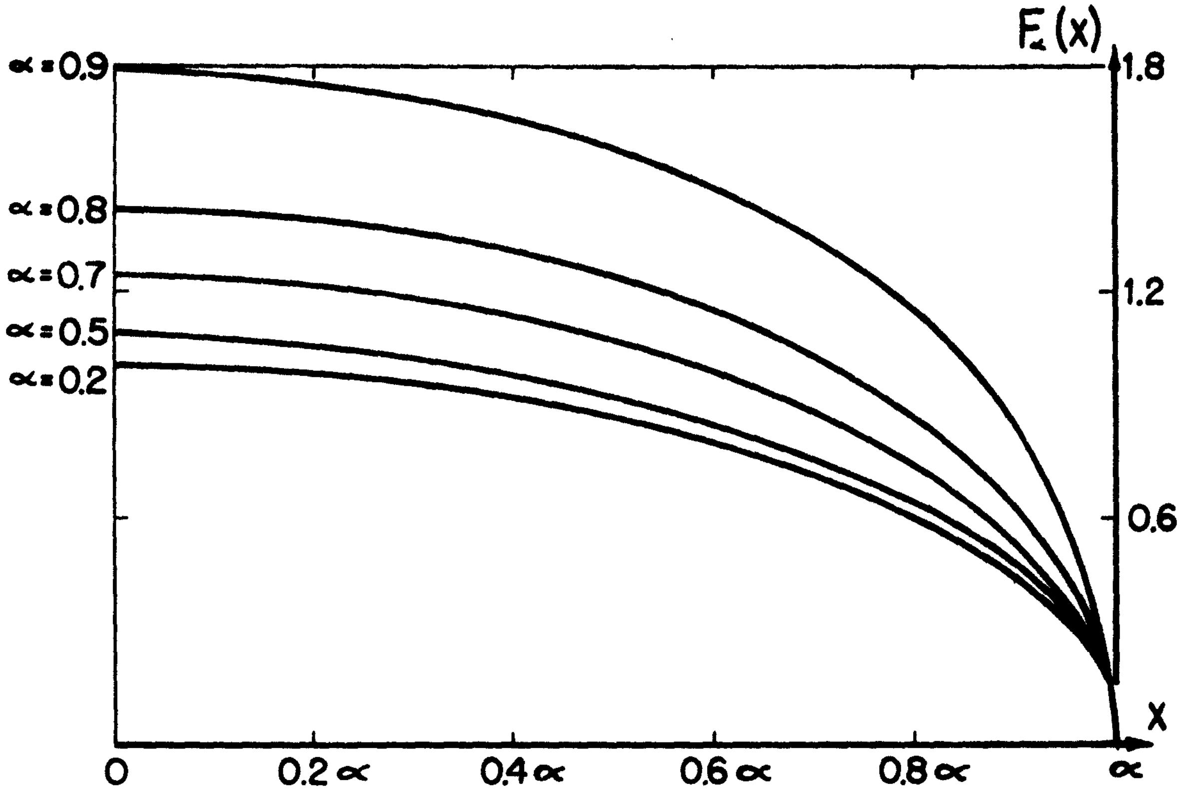

Figure 14

Fig. 14 The variation with $\underline{x}$ and α of the function $F_\alpha(x)$ for constant p_o and $\eta = 0.25$.

Table 7: The variation of Ω_2, Π_2 and n_2 with a/c. ($\eta = 0.25$).

a/c	0.2	0.3	0.5	0.7	0.8	0.9
Ω_2	0.998	0.993	0.965	0.891	0.822	0.707
Π_2	0.2	0.7	3.5	10.9	17.8	29.3
n_2	0.5	1.3	7.2	25.9	47.9	100.2

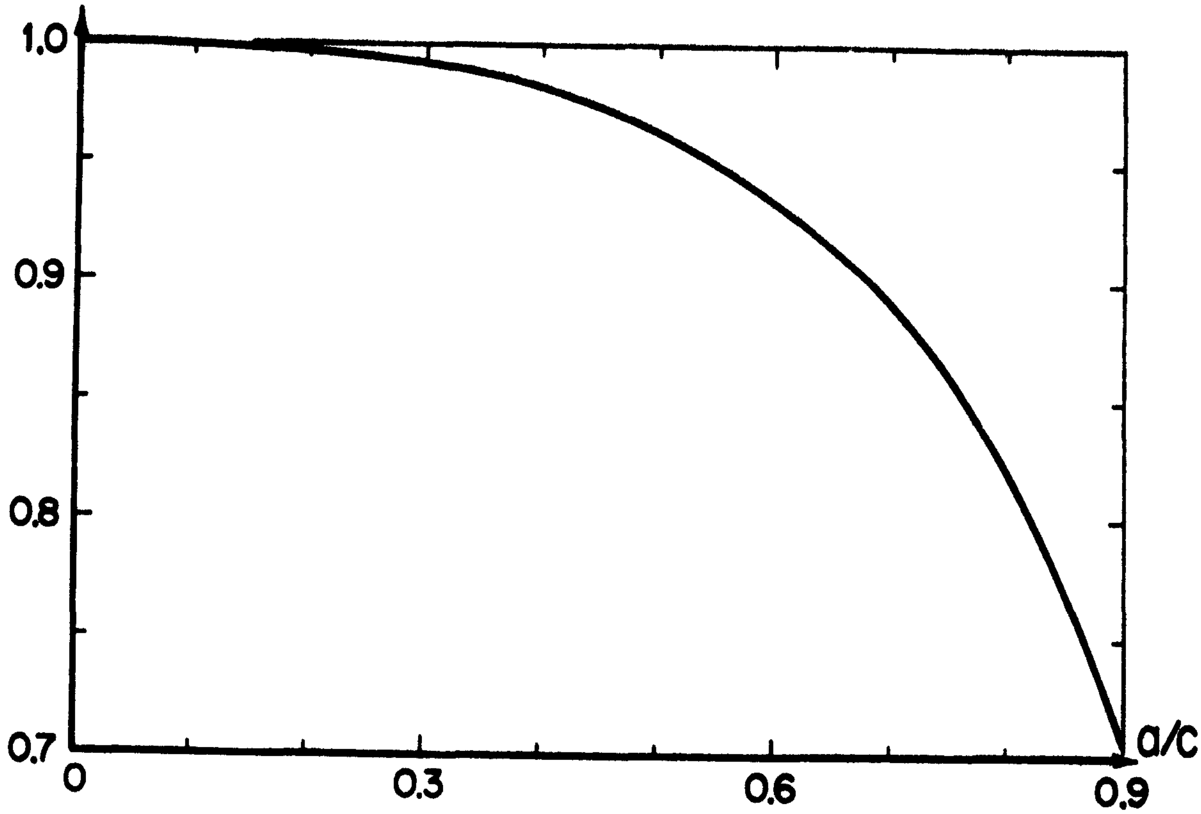

Figure 15

Fig. 15 The variation with $p_{cr}/p_{cr}^{(\infty)}$ with a/c in the case $\eta = 0.25$ and for constant pressure on the crack surface.

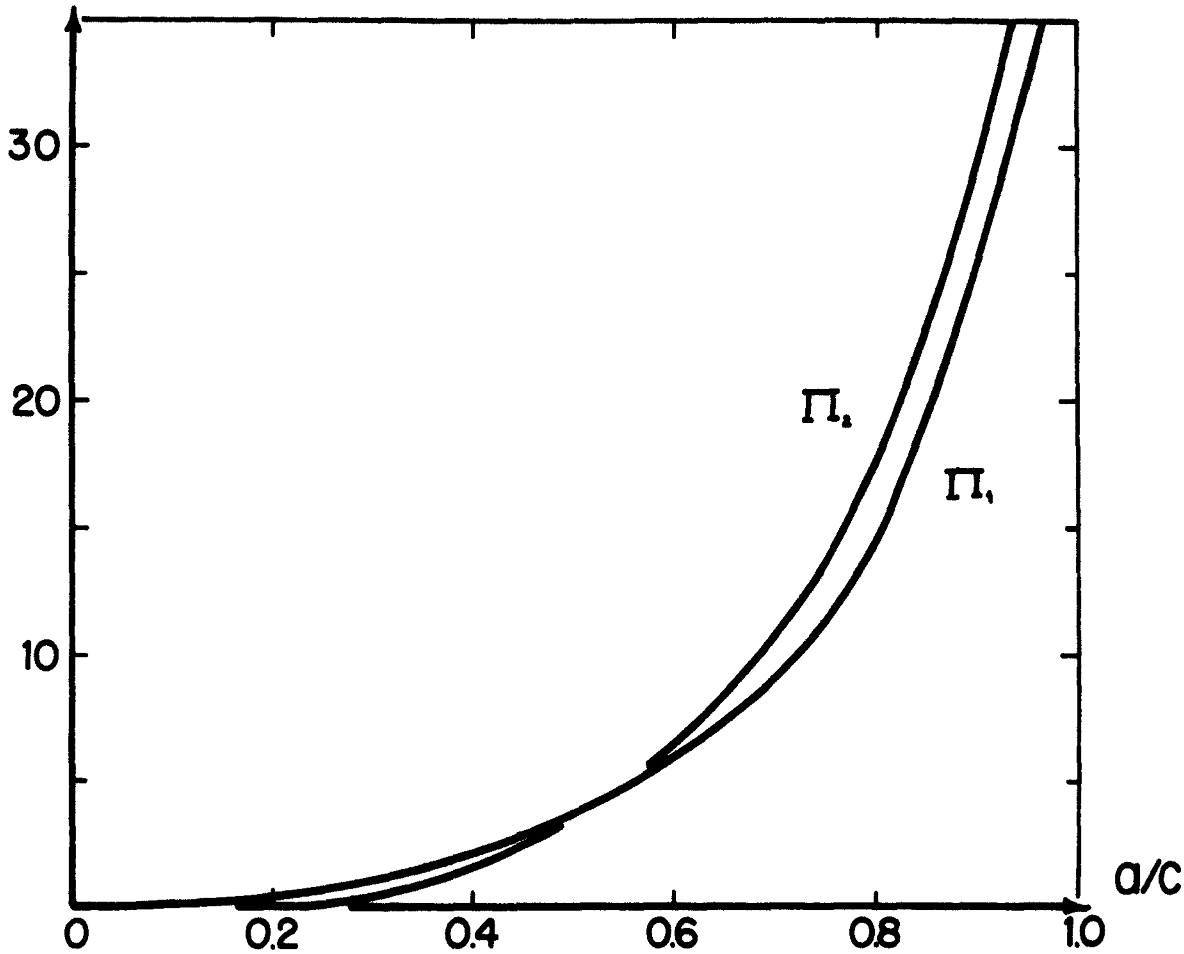

Figure 16

Fig. 16 The variation with a/c of the percentage increase in the critical pressure for constant pressure on the crack surfaces. The curve π_1 corresponds to the case in which $u_\rho = 0$ on the cylindrical surface and is independent of η; the curve π_2 corresponds to the case in which $\sigma_{\rho\rho} = 0$ on the cylindrical surface and $\eta = 0.25$.

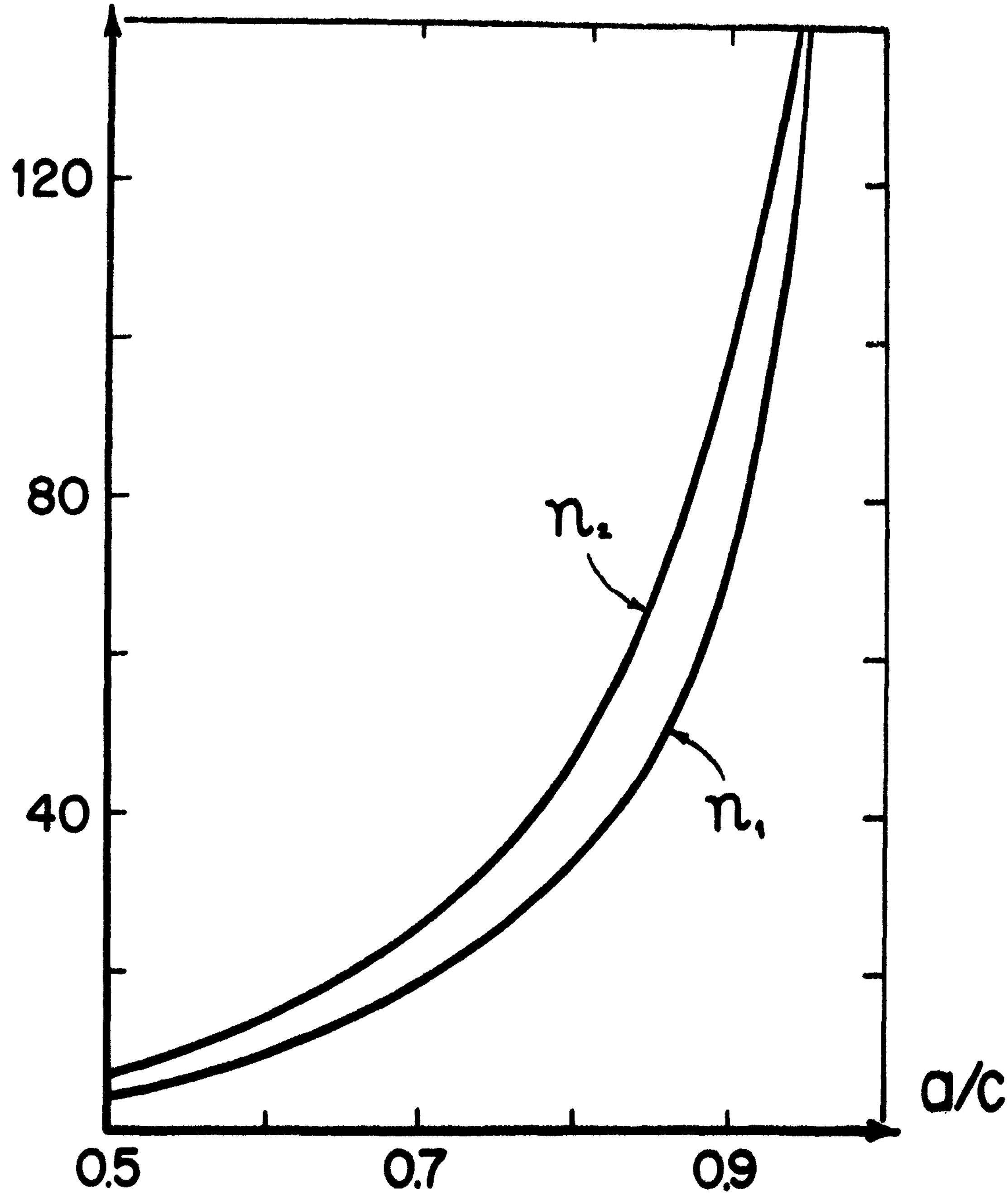

Figure 17

Fig. 17 The variation with a/c of the percentage increase in the stress intensity factor for constant pressure on the crack surfaces. The curve n_1 corresponds to the case in which $u_\rho = 0$ on the cylindrical surface and is independent of η; the curve n_2 corresponds to the case in which $\sigma_{\rho\rho} = 0$ on the cylindrical surface and $\eta = 0.25$.

pronounced in the case in which the radial component of the surface displacement of the cylinder is zero.

3.6. Solution of Problem (b) in the Case of a Variable Internal Pressure.

To illustrate the effect of the application of a variable pressure to the surfaces of the crack the numerical calculations were repeated for the case in which

$$f(\rho) = \frac{p_o}{2\mu}\left(1 - \frac{\rho^2}{c^2}\right), \quad 0 \leqslant \rho < 1. \tag{3.93}$$

In this case the free term of the integral equation (3.68) assumes the form

$$h(t) = \frac{p_o c}{\pi\mu}\left(\frac{t}{c} - \frac{2t^3}{c^3}\right). \tag{3.94}$$

If we insert this value for h(t) into equation (3.68) and make the transformations (3.81) we find that the function $\phi(t)$ is a solution of the integral equation

$$\phi(t) + \int_0^\alpha K(t, u)\,\phi(u)du = t - \tfrac{2}{3}t^3 \tag{3.95}$$

where the kernel K(t, u) is given by the equation (3.83).

Numerical solutions of this equation were obtained for the four values 0.2, 0.5, 0.7 and 0.9 of $\alpha = a/c$ and for $\eta = 0.25$. The results are given in Table 8 and are illustrated graphically in Fig. 18. The function $F_\alpha(x)$ defined by equation (3.89) was also calculated for these four values of α. The results are shown in Table 9 and graphically in Fig. 19. Here the upper curves show the distribution of pressure on the crack surface and the lower curves give the resulting shape of the crack; the length b, the depth of the crack in the case a << c, is defined by equation (3.90).

The quantities Ω_2 and n_2 which are defined by equations (3.86) and (3.92) and from which can be derived the critical pressure and the percentage increase in the stress intensity factor, were calculated for these four values of the ratio a/c. The results are shown in Table 10. In comparing these results with those corresponding to constant internal

Table 8: Values of $\phi(r\alpha/12)$, the solution of the integral equation (3.95) for four values of α.

$r \setminus \alpha$	0.2	0.5	0.7	0.9
1			0.069	0.110
2	0.033	0.088	0.137	0.219
3			0.204	0.325
4	0.067	0.175	0.269	0.430
5			0.330	0.525
6	0.100	0.255	0.389	0.621
7			0.440	0.700
8	0.132	0.331	0.491	0.791
9			0.527	0.840
10	0.164	0.392	0.570	0.930
11			0.588	0.949
12	0.195	0.500	0.617	1.002

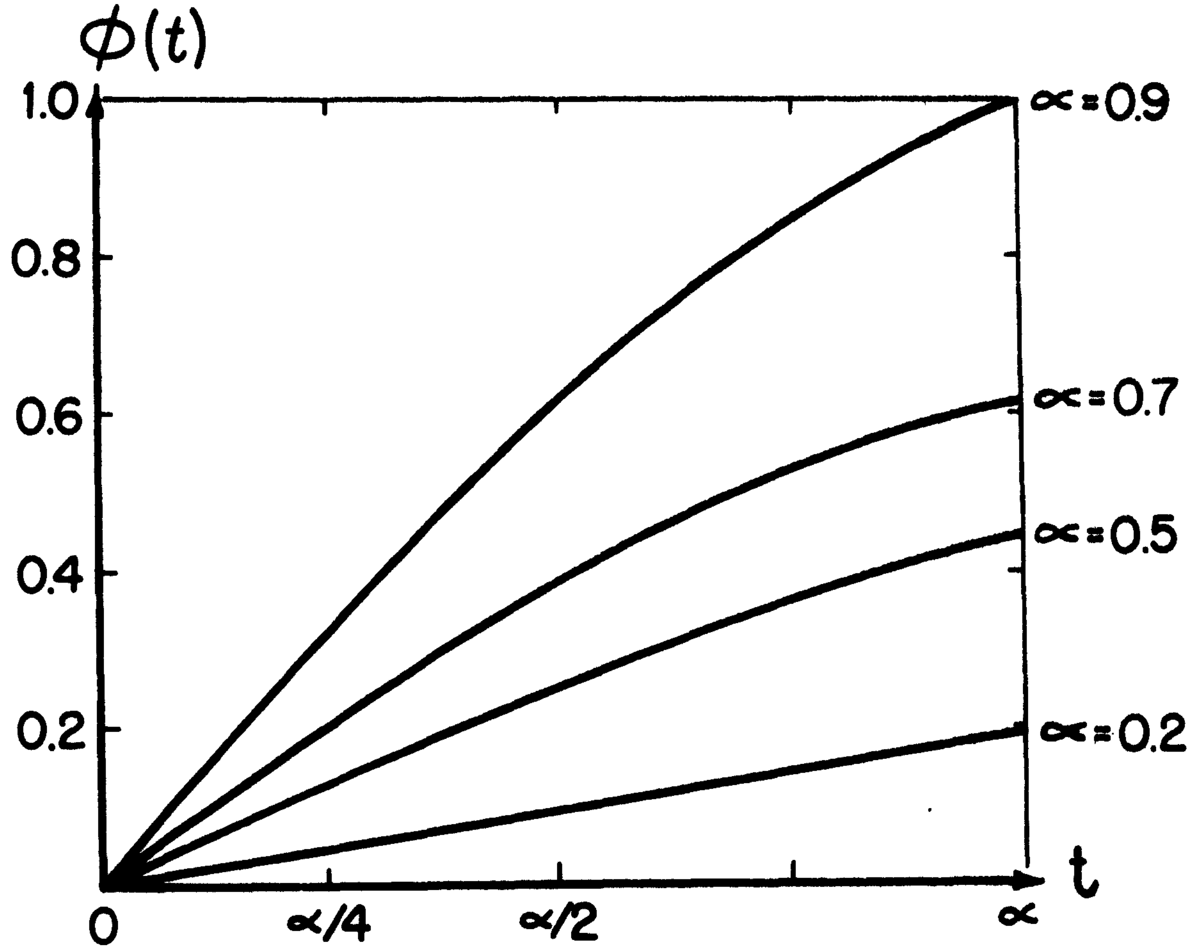

Figure 18

Fig. 18 The variation with t and α of the function $\Phi(t)$ for a pressure $p_o(1-\rho^2/c^2)$ on the crack surfaces and $\eta = 0.25$.

Table 9: Values of the function $F_{\alpha}(r\alpha/12)$ defined by equations (3.89) and (3.95) for four values of α.

$r \setminus \alpha$	0.2	0.5	0.7	0.9
0	0.995	1.008	1.082	1.347
1			1.077	1.341
2	0.981	0.991	1.061	1.321
3			1.035	1.289
4	0.936	0.938	0.998	1.243
5			0.951	1.185
6	0.858	0.848	0.892	1.112
7			0.821	1.026
8	0.736	0.712	0.737	0.923
9			0.638	0.801
10	0.543	0.512	0.518	0.653
11			0.361	0.453
12	0.000	0.000	0.000	0.000

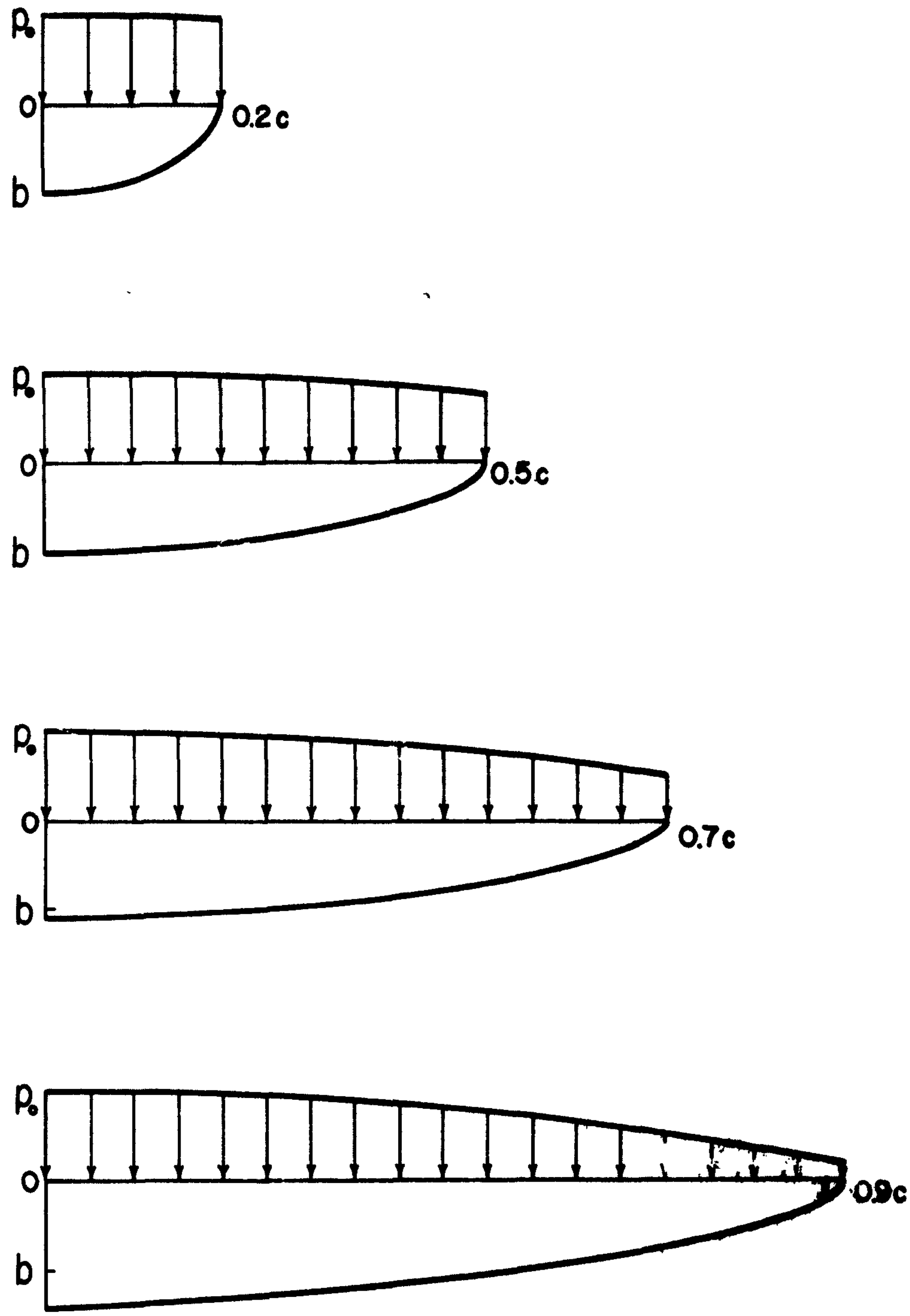

Figure 19

Fig. 19 The shape of the crack for four values of a/c when the pressure is $p_o(1-\rho^2/c^2)$ and $\eta = 0.25$. The upper curves show the distribution of pressure on the crack surface and the lower curves the corresponding shape of the crack. [$\underline{b} = 4(1-\eta^2)p_oa/\pi E$).

Table 10: The variation of the functions Ω_2, Π_2 and n_2 with a/c when the internal pressure is $p_0(1 - \rho^2/c^2)$. ($\eta = 0.25$).

a/c	0.2	0.5	0.7	0.9
Ω_2	1.011	1.055	1.065	0.948
n_2	1.5	20	31	142

pressure (Table 7 above) it should be noted that in the second case the average pressure on the crack is

$$2p_0 a^{-2} \int_0^a \rho(1-\rho^2/c^2)\,d\rho = p_0\left[1-\tfrac{1}{2}(a/c)^2\right]$$

so that if a is nearly equal to c the average pressure is appreciably less than p_0.

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